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The geometry of the full Kostant–Toda lattice of $sl(4, \mathbf{C})$

Barbara A. Shipman

Department of Mathematics, The University of Texas at Arlington, Box 19408, Arlington, TX 76019-0408, USA

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Abstract

The full Kostant–Toda lattice is an integrable system whose geometry appears naturally in the setting of flag manifolds but is not easily apparent in the original phase space. Separatrices in the flows for two different families of integrals in the Toda lattice of $sl(4, \mathbf{C})$ appear in splittings of weight polytopes associated to partial flag manifolds of $Sl(4, \mathbf{C})$. Each family of integrals gives rise to a $\mathbf{C}^*$-bundle of level set varieties with singular fibers. In a neighborhood of a separatrix, the monodromy is a single twist of the noncompact cycle around the cylinder, $\mathbf{C}^*$. Near another type of singularity, the cycle is twisted twice; this interchanges the source and sink of one of the Hamiltonian flows. ©2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

The Toda lattice, in its original form, is a Hamiltonian system for n particles on a line with constant of motion

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \sum_{i=1}^{n-1} e^{q_i - q_{i+1}}.$$

It evolves on the subset of R^n with $\sum_{i=1}^{n} p_i = 0$ and q_n constant. Through a change of variables [4] and a transformation of matrices [7], the Toda lattice takes on a simple form. With $a_i = e^{q_i - q_{i+1}}$, $i = 1, \dots, n-1$, and $b_i = -p_i$, $i = 1, \dots n$, let

$$X = \begin{pmatrix} b_1 & 1 & 0 & \cdots & 0 \\ a_1 & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & a_{n-1} & b_n \end{pmatrix}.$$

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Then Hamilton's equations assume the Lax form

$$\dot{X}(t) = [X(t), \Pi_{\mathcal{N}_-} X(t)], \tag{1}$$

where $\Pi_{\mathcal{N}_-} X$ is the strictly lower triangular part of X, and the Hamiltonian is $(1/2)\mathrm{tr}X^2$.

The solution of (1) with initial condition X_0 is obtained by factoring e^{tX_0} as

$$e^{tX_0} = n(t)b(t), \tag{2}$$

where $n(t)$ is lower unipotent and $b(t)$ is upper triangular, and then conjugating X_0 by the lower unipotent factor:

$$X(t) = n^{-1}(t)X_0 n(t). \tag{3}$$

The symmetric functions $(1/k)\mathrm{tr}X^k$, with $k = 2, \ldots, n$, are constants of motion. For $k > 2$, the Hamiltonian vector field is obtained by replacing $\Pi_{\mathcal{N}_-} X(t)$ in (1) by $\Pi_{\mathcal{N}_-} X^{k-1}(t)$. The flow is found by factoring $e^{tX_0^{k-1}}$ as before and conjugating X_0 by the lower unipotent factor as in (3). These flows commute, and the system is completely integrable [3].

The full Kostant–Toda lattice is a generalization of this in two directions. First, there is no restriction on the entries of X below the diagonal, and second, the entries on (and below) the diagonal are complex-valued. X belongs to $\epsilon + \mathcal{B}_-$, where $\mathcal{B}_-$ is the lower triangular subalgebra of $sl(n, \mathbf{C})$ and ϵ the matrix with 1's on the superdiagonal and 0's elsewhere; $\epsilon + \mathcal{B}_-$ is a Poisson manifold foliated by complex symplectic leaves of different dimensions.

On a generic leaf, (1) remains completely integrable. This is shown in [2,3], where the constants of motion are given explicitly. In addition to the $(1/k)\mathrm{tr}X^k$, there are constants of motion that are rational in the entries of X; the level sets of these integrals are not compact, and the flows are not complete. In [2] the authors study the geometry of a generic level set by embedding it into the flag manifold of $Sl(n, \mathbf{C})$. Here the flows are completed and the level set becomes a union of orbits of the diagonal complex torus.

One can now bring in the moment map, which sends the flag manifold to the weight polytope of an irreducible representation of $sl(n, \mathbf{C})$. The image of the closure of a torus orbit is the convex hull of a subset of the vertices; this may be a single vertex, a lower-dimensional face, or a polytope with the same dimension as the full polytope. In particular, the fixed points of the torus correspond to the vertices and the generic orbits to the complete polytope.

In $sl(4, \mathbf{C})$, $\epsilon + \mathcal{B}_-$ has dimension 9. A generic symplectic leaf is 8-dimensional, and there are three functions $(1/k)\mathrm{tr}X^k$. There are two integrals, I and J, each of which completes this family. Neither of these is constant along the vector field generated by the other, leaving two distinct families of integrals. In the flag manifold, I is defined purely in terms of the manifold of partial flags $\{V^1 \subset V^3 \subset \mathbf{C}^4\}$, while J is defined on the complementary Grassmannian, $\{V^2 \subset \mathbf{C}^4\}$.

In each case, a typical level set of the integrals is essentially a family of 3-dimensional torus orbits parametrized by $\mathbf{P}^1$. The flows for the $(1/k)\mathrm{tr}X^k$ generate the torus action, while the flow for the remaining integral generates the 1-parameter family. A level set of I typically intersects a level set of J in two distinct orbits. This is a reflection of an involution in the Toda lattice that comes from the $\mathbf{Z}_2$-symmetry in the Dynkin diagram of $sl(n, \mathbf{C})$ [10]. The involution preserves the constants of motion and interchanges the two orbits.

The projection of a level set of I to the partial flag manifold $\{V^1 \subset V^3 \subset \mathbf{C}^4\}$ is typically the closure of a single generic torus orbit [3]. We will show that this is also true if I is replaced by J and $\{V^1 \subset V^3 \subset \mathbf{C}^4\}$ by the Grassmannian $\{V^2 \subset \mathbf{C}^4\}$. In both cases, the degenerate level sets correspond to splittings of the moment polytope along interior faces.

In the full flag manifold, the $\mathbf{P}^1$ of $(\mathbf{C}^*)^3$-orbits in a typical level set of I contains two orbits whose points are fixed by the I-flow; they do not belong to the generic leaf. Removing these orbits yields a variety generated by a $(\mathbf{C}^*)^4$-action — the $(\mathbf{C}^*)^3$-torus of the basic flows together with a $\mathbf{C}^*$-action of the completed I-flow. Taking the quotient by the torus action now yields a $\mathbf{C}^*$-fiber bundle with singular fibers. The monodromy near the singularities is studied by measuring the twisting of the noncompact cycle in the generic fiber as the value of I traverses a loop about the singular value. At a degenerate level set, the fiber consists in two intersecting copies of $\mathbf{C}$, and the cycle is twisted once around the $\mathbf{C}^*$-cylinder. Where the two fixed points of the I-flow coincide, the fiber is $\mathbf{C}$, and the cycle is twisted twice. The latter monodromy is found in the $sl(2,\mathbf{C})$ Toda lattice near its unique singular level set [9]; it reflects a reversal of the two fixed points of the I-flow, one a source and the other a sink. Both types of monodromy also occur in the $\mathbf{C}^*$-bundle of the J-fibration.

2. Background

2.1. The full Kostant–Toda lattice

The Poisson structure on $\epsilon + \mathcal{B}_-$ is defined through the decomposition of $sl(n,\mathbf{C})$ into its upper triangular and lower nilpotent subalgebras:

$$sl(n,\mathbf{C}) = \mathcal{B}_+ \oplus \mathcal{N}_-.$$

The Killing form $\langle X, Y\rangle = 2n \cdot \mathrm{tr}(XY)$ on $sl(n,\mathbf{C})$ identifies the dual, $\mathcal{B}_+^*$, of $\mathcal{B}_+$ with $\mathcal{N}_-^\perp$, which is $\mathcal{B}_-$. This gives an identification of $\epsilon + \mathcal{B}_-$ with the dual of a Lie algebra:

$$\epsilon + \mathcal{B}_- \cong \mathcal{B}_- = \mathcal{N}_-^\perp \cong \mathcal{B}_+^*. \tag{4}$$

Through this identification, $\epsilon + \mathcal{B}_-$ acquires a Poisson structure from the Lie–Poisson structure defined on $\mathcal{B}_+^*$ by

$$\{f, g\}(\beta) = \beta([\nabla f(\beta), \nabla g(\beta)]),$$

where the gradient $\nabla f(\beta)$ belongs to $\mathcal{B}_+$: for $\gamma \in \mathcal{B}_+^*$,

$$\gamma(\nabla f(\beta)) = \lim_{h\to 0} \frac{f(\beta + h\gamma) - f(\beta)}{h}.$$

The symplectic leaves in $\mathcal{B}_+^*$ are the coadjoint orbits of B_+, the Lie group of $\mathcal{B}_+$. Through (4), these leaves are identified with the orbits of an action of B_+ on $\epsilon + \mathcal{B}_-$:

$$Ad_b^* X = \epsilon + \Pi_{\mathcal{B}_-}(b^{-1}(X - \epsilon)b),$$

where $\Pi_{\mathcal{B}_-}$ is projection onto the lower triangular part. The Lax equation (1) is generated by the Hamiltonian $(1/2)\mathrm{tr}X^2$. Here, however, the dimension of a leaf is greater than $n-1$, and there are constants of motion in addition to the $(1/k)\mathrm{tr}X^k$. To obtain a completely integrable system on a generic symplectic leaf, one must find Casimirs to cut out the leaf and sufficiently many additional integrals.

Proposition 2.1. (Ref. [2,3]) *For* $k = 0, \dots, [(n-1)/2]$, *denoted by* $(X - \lambda Id)_{(k)}$ *the result of removing the first k rows and last k columns from* $X - \lambda Id$, *and let* λ_{rk}, $r = 1, \dots, n-2k$, *denote the roots of*

$$\tilde{Q}_k(X, \lambda) = \det(X - \lambda Id)_{(k)} = E_{0k}\lambda^{n-2k} + \cdots + E_{n-2k,k}.$$

The λ_{rk} *are constants of motion for the full Kostant–Toda lattice. The coefficients of the monic polynomial*

$$Q_k(X, \lambda) = \frac{\det(X - \lambda Id)_{(k)}}{E_{0k}} = \lambda^{n-2k} + I_{1k}\lambda^{n-2k-1} + \cdots + I_{n-2k,k}$$

are constants of motion equivalent to the λ_{rk}. *They are called the k-chop integrals. (The* I_{r0} *are the coefficients of the characteristic polynomial of X.) The functions* $I_{1k} = \sum_r \lambda_{rk}$ *are Casimirs on* $\epsilon + \mathcal{B}_-$, *and the* I_{rk} *for* $r > 1$ *constitute a complete involutive family of integrals for the generic symplectic leaves of* $\epsilon + \mathcal{B}_-$ *cut out by the Casimirs* I_{1k}.

The symplectic leaves are cut out by functions that are rational in the entries of X; they are not compact, and the Hamiltonian flows of the integrals are not complete. These integrals may be computed as in the following proposition.

Proposition 2.2. (Ref. [3]) *Choose* $X \in \epsilon + \mathcal{B}_-$, *and break it into blocks of the indicated sizes as*

$$X = \begin{array}{c} k \\ n-2k \\ k \end{array} \overset{\begin{array}{ccc} k & n-2k & k \end{array}}{\begin{pmatrix} X_1 & X_2 & X_3 \\ X_4 & X_5 & X_6 \\ X_7 & X_8 & X_9 \end{pmatrix}}$$

where k is an integer such that $0 \le k \le [(n-1)/2]$. *If* $\det X_7 \neq 0$, *define the matrix* $\phi_k(X)$ *by*

$$\phi_k(X) = X_5 - X_4 X_7^{-1} X_8 \in Gl(n-2k, \mathbf{C}), \quad k \neq 0,$$

$$\phi_0(X) = X.$$

The I_{rk} *are the coefficients of the polynomial* $det(\lambda - \phi_k(X))$.

The generic level sets of these constants of motion are studied in [3] using the geometry of generalized flag manifolds. This is made possible through the following fact.

Proposition 2.3. (Ref. [6]) *Let* $\lambda^n - s_2\lambda^{n-2} - \cdots - s_n$ *be the characteristic polynomial of* $X \in \epsilon + \mathcal{B}_-$, *and consider the companion matrix,*

$$C = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & \ddots & 1 \\ s_n & \cdots & \cdots & s_2 & 0 \end{pmatrix}.$$

There exists a unique lower unipotent matrix L such that $X = LCL^{-1}$.

In the generic case, in which the eigenvalues λ_i are distinct, $C = V\Lambda V^{-1}$, where V is a Vandermonde matrix and $\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$, so that

$$X = LV\Lambda V^{-1}L^{-1}. \tag{5}$$

Let $(\epsilon + \mathcal{B}_-)_\Lambda$ denote the subset of $\epsilon + \mathcal{B}_-$ with spectrum Λ. The uniqueness of L implies that the mapping

$$\Psi_\Lambda : (\epsilon + \mathcal{B}_-)_\Lambda \to Sl(n, \mathbf{C})/B, \qquad X \mapsto V^{-1}L^{-1} \mathrm{mod}\, B,$$

is an embedding. Its image is dense in $Sl(n, \mathbf{C})/B$. Under this embedding, the Toda flow generated by $(1/k)\mathrm{tr}X^k$ becomes the flow $e^{t\Lambda^{k-1}}V^{-1}L^{-1}$ mod B in the flag manifold. Together, the flows for $k = 2, \dots, n$ generate the action of the diagonal complex torus $(\mathbf{C}^*)^{n-1}$ so that in $Sl(n, \mathbf{C})/B$, each compactified level set of the constants of motion is a union of torus orbits. The mapping Ψ_Λ is called the *torus embedding*.

2.2. *The moment map*

All facts about moment maps referred to here may be found in [5]. This section briefly reviews the main theorem and fixes some notation.

Let G be a complex semisimple Lie group, H a Cartan subgroup of G, and B a Borel subgroup containing H. Denote by $\mathcal{H}$ the Lie algebra of H and by $\mathcal{H}^*$ the dual of $\mathcal{H}$. Given a parabolic subgroup P of G containing B, the homogeneous space G/P may be realized as the orbit of G through a projectivized highest weight vector in the projectivization, $P(V)$, of an irreducible representation $\rho : G \to Gl(V)$.

Let $\mathcal{A}$ be the set of weights of ρ taken with multiplicity, and choose a basis of weight vectors $\{v_\alpha : \alpha \in \mathcal{A}\}$ for V. A point $[X]$ in $P(V)$, represented by X in V, determines uniquely, up to a common scalar factor, the collection of numbers $\pi_\alpha(X)$, where $X = \sum_{\alpha\in\mathcal{A}} \pi_\alpha(X)v_\alpha$. In terms of the homogeneous *Plücker coordinates* $[\pi_\alpha(X)]$, the moment map has the expression

$$\mu : G/P \to \mathcal{H}^*_{\mathbf{R}}, \qquad [X] \mapsto \frac{\sum_{\alpha\in\mathcal{A}} |\pi_\alpha(X)|^2 \alpha}{\sum_{\alpha\in\mathcal{A}} |\pi_\alpha(X)|^2},$$

where $\mathcal{H}^*_{\mathbf{R}}$ is the real part of $\mathcal{H}^*$. Its image is the weight polytope of V, also referred to as the *moment polytope* of G/P. It is the convex hull in $\mathcal{H}^*_{\mathbf{R}}$ of the weights of ρ.

Consider the orbit $W \cdot \alpha_0$, where W is the Weyl group of G and α_0 is the highest weight of V determined by the choice of B. The weight vectors v_α such that $[v_\alpha] \in G/P$ are exactly those for which $\alpha \in W \cdot \alpha_0$. These are the fixed points of H in G/P, and their images under μ are the vertices of the weight polytope: $\mu([v_\alpha]) = \alpha$.

The moment map sends each torus orbit to the (open) polytope whose vertices are those that correspond to the fixed points in the closure of the orbit. The complex dimension of a torus orbit is equal to the real dimension of its image. The moment map induces a partition of G/P into equivalence classes called strata, where each stratum is the union of all torus orbits that correspond to the same convex polytope. The image of the generic stratum, on which no π_α vanishes, is the complete moment polytope.

2.3. $G = Sl(4, \mathbf{C})$

2.3.1. *The flag manifolds and moment maps*

Let $\mathcal{H}$ be the diagonal Cartan subalgebra of $sl(4, \mathbf{C})$, and denote by L_i the linear function in $\mathcal{H}^*$ that sends an element of $\mathcal{H}$ to its ith diagonal entry. $\mathcal{H}^*$ is the quotient space

$$\mathcal{H}^* = \frac{\left\{\sum_{i=1}^4 c_i L_i : c_i \in \mathbf{C}\right\}}{\left\langle \sum_{i=1}^4 L_i \right\rangle} \cong \mathbf{C}^3.$$

The weight lattice of $sl(4, \mathbf{C})$ is the integer lattice generated by the L_i.

Let S be the standard representation, $\mathbf{C}^4$, and denote by $[\pi_1 : \cdots : \pi_4]$, or simply by $[\pi_i]$, the Plücker coordinates on $P(S)$ with respect to the standard basis vectors e_i. The dual, $S^* = \wedge^3 S$, has weight basis $\{e_i^*\}$, where e_i^* satisfies $e_i \wedge e_i^* = e_1 \wedge \cdots \wedge e_4$; $[\pi_i^*]$ denotes the Plücker coordinates on $P(S^*)$ with respect to this basis. The weights of e_i and e_i^* are L_i and $-L_i$, respectively. The Plücker coordinates on $P(\wedge^2 S)$ with respect to the weight basis $\{e_i \wedge e_j\}_{i<j}$ are denoted as $[\pi_{12} : \pi_{13} : \pi_{14} : \pi_{23} : \pi_{24} : \pi_{34}]$, where π_{ij} corresponds to $e_i \wedge e_j$, with weight $L_i + L_j$.

Given an element of $Sl(4, \mathbf{C})$, its first n columns span an n-dimensional subspace, V^n, of $\mathbf{C}^4$, which determines a flag $V^1 \subset V^2 \subset V^3$. Two matrices determine the same flag if and only if they belong to the same coset in the quotient G/B, where $G = Sl(4, \mathbf{C})$ and B is the upper triangular subgroup. The flag manifold, G/B, is realized as the orbit of G through the projectivized highest weight vector $[e_1 \otimes (e_1 \wedge e_2) \otimes (e_1 \wedge e_2 \wedge e_3)]$ in $P(\Gamma_{G/B})$, where $\Gamma_{G/B}$ is the irreducible representation of G with highest weight $3L_1 + 2L_2 + L_3$. It is embedded in the product of the projectivized fundamental representations by the mapping

$$\begin{aligned} &G/B \to P(S) \times P(\wedge^2 S) \times P(S^*), \\ &[ge_1 \otimes (ge_1 \wedge ge_2) \otimes (ge_1 \wedge ge_2 \wedge ge_3)] \mapsto [ge_1] \\ &\qquad \times [ge_1 \wedge ge_2] \times [ge_1 \wedge ge_2 \wedge ge_3], \end{aligned}$$

where $g \in G$.

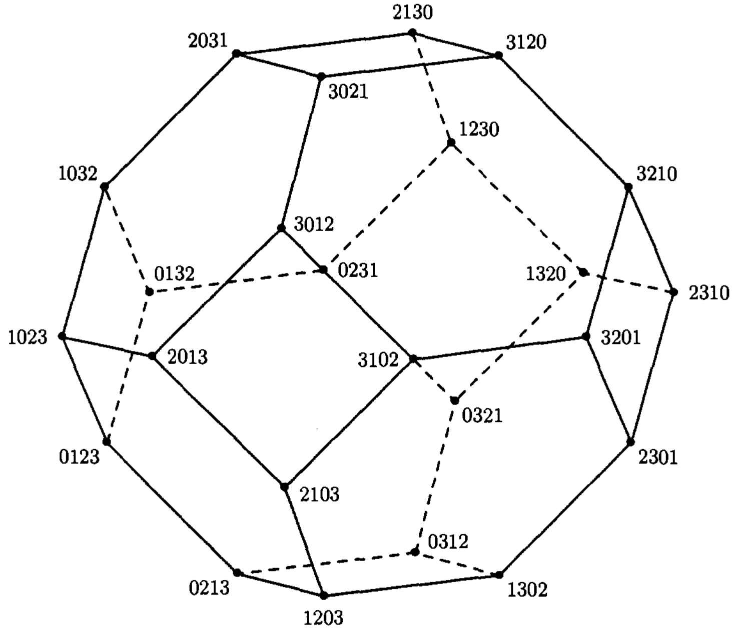


Fig. 1. The moment polytope, $\diamond$, of the flag manifold $Sl(4, \mathbf{C})/B$. The vertices are $3L_i + 2L_j + L_k$ with i, j, k distinct; they are labeled by the coefficients of L_1, L_2, L_3 and L_4, in that order.

In terms of the Plücker coordinates on the product space, the moment map on G/B is the sum of the moment maps on the three factors:

$$\mu\left[\left(\sum \pi_i e_i\right) \otimes \left(\sum \pi_{ij} e_i \wedge e_j\right) \otimes \left(\sum \pi_i^* e_i^*\right)\right]$$

$$= \frac{\sum |\pi_i|^2 L_i}{\sum |\pi_i|^2} + \frac{\sum |\pi_{ij}|^2 (L_i + L_j)}{\sum |\pi_{ij}|^2} + \frac{\sum |\pi_i^*|^2 (-L_i)}{\sum |\pi_i^*|^2}.$$

Its image, denoted $\diamond$, is the weight polytope of $\Gamma_{G/B}$ (Fig. 1). The vertices constitute the orbit of the Weyl group, Σ_4, through the highest weight $3L_1 + 2L_2 + L_3$.

The image of the natural projection of G/B to $P(S) \times P(S^*)$ is the manifold of partial flags $V^1 \subset V^3$. It is realized as G/P_1, where P_1 is the subgroup with 0's below the diagonal in the first column and to the left of the diagonal in the last row. G/P_1 is the orbit of G through $[e_1 \otimes (e_1 \wedge e_2 \wedge e_3)]$ in the projectivization of the adjoint representation of $sl(4, \mathbf{C})$. Its moment polytope, denoted $\triangle$, is shown in Fig. 2. The vertices of $\triangle$ are the roots $L_i - L_j$ of $sl(4, \mathbf{C})$, or equivalently, the weights $2L_i + L_j + L_k$, with i, j, k distinct.

The projection of G/B to $P(\wedge^2 S)$ is the Grassmannian of 2-planes, $Gr(2, 4)$. It is the orbit of G through $[e_1 \wedge e_2]$ in $P(\wedge^2 S)$, and its moment polytope is the octahedron with vertices $L_i + L_j$, in Fig. 3.

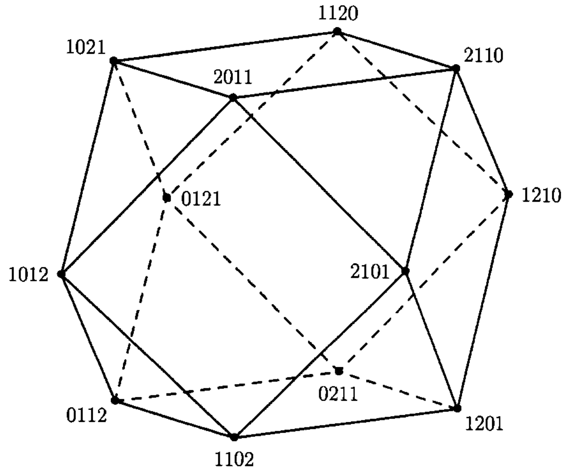


Fig. 2. The moment polytope, Δ, of the partial flag manifold $\{V^1 \subset V^3 \subset \mathbf{C}^4\}$. The vertices are $2L_i + L_j + L_k$ with i, j, k distinct. The vertex $2L_1 + L_2 + L_3$ is labeled 2110, and the others are labeled similarly.

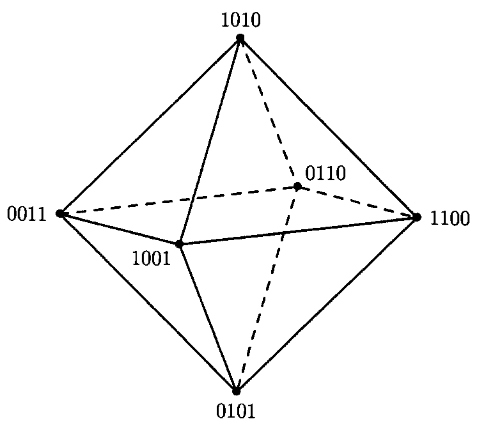


Fig. 3. The moment polytope, $\square$, of the Grassmannian $Gr(2,4)$. The vertices are $L_i + L_j, i \neq j$, labeled by the coefficients of L_1, L_2, L_3 and L_4, in that order.

2.3.2. *Splittings of the polytopes*

At each fixed point of the torus action in G/P_1, exactly one π_i and one π_j^* does not vanish. Those where $\pi_i \neq 0$ correspond to the vertices of a triangular face, and those where $\pi_i^* \neq 0$ to the vertices of the opposite face. The polytope of a 3-dimensional orbit where π_i or π_i^* is the only vanishing coordinate is missing the vertices of the corresponding triangular face. These polytopes are denoted $\Delta(\backslash i)$ and $\Delta(\backslash i^*)$, respectively. They are obtained by splitting Δ along an interior hexagon as in Fig. 4. At a fixed point in $Gr(2, 4)$, exactly one π_{ij} does not vanish. The polytope of an orbit where π_{ij} is the only vanishing coordinate is a half-octahedron, denoted as $\square(\backslash ij)$ (Fig. 5).

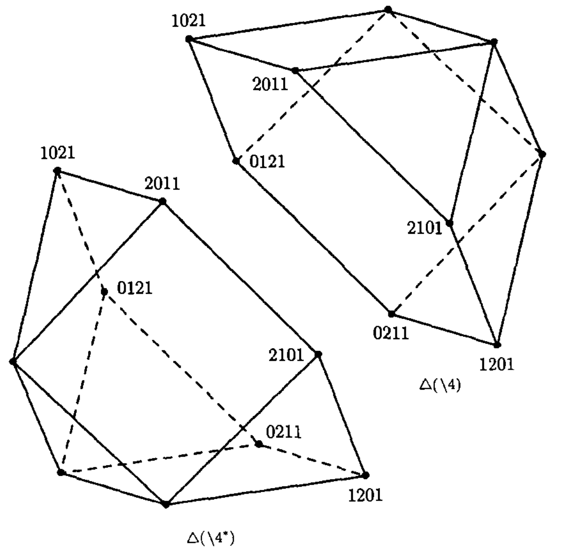


Fig. 4. A splitting of Δ along an interior hexagon.

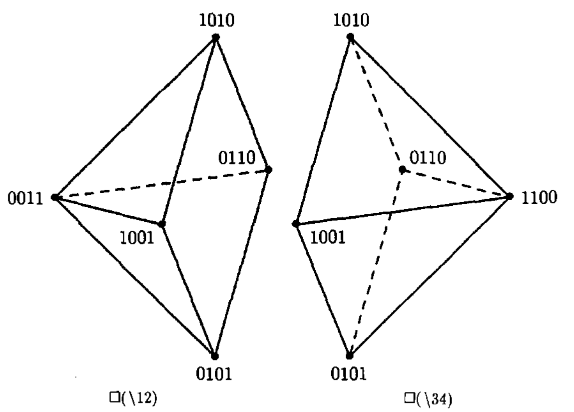
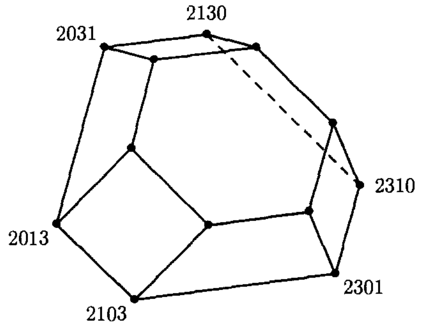
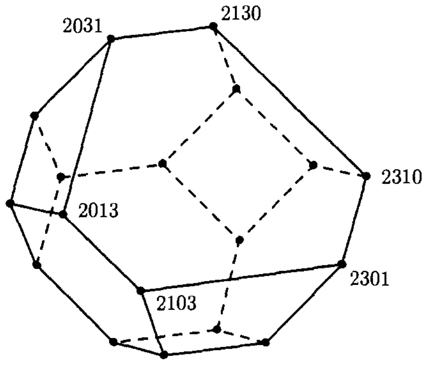


Fig. 5. A splitting of □ along an interior square.

At each fixed point in G/B, exactly one π_i, one π_j^*, and one π_{ik} does not vanish. The polytope of a torus orbit on which π_i is the only vanishing coordinate is missing the vertices of a hexagonal face. It is denoted as $\diamond(\backslash i)$ and is the larger of the two polytopes obtained by splitting $\diamond$ along an interior hexagon as in Fig. 6; its complement is denoted $\diamond(\backslash i)^{\mathrm{c}}$. If only π_i^* vanishes, then the vertices of the opposite hexagonal face are missing, and $\diamond$ is split along the interior hexagon parallel to this, producing the polytopes $\diamond(\backslash i^*)$ and $\diamond(\backslash i^*)^{\mathrm{c}}$. The polytope of an orbit where π_{ij} and no other coordinate vanishes is denoted $\diamond(\backslash ij)$ and is obtained by splitting $\diamond$ along an interior square (Fig. 7).



$\diamond(\backslash 1)^c$



$\diamond(\backslash 1)$

Fig. 6. A splitting of $\diamond$ along an interior hexagon.

Notation. A polytope that occurs as the image of a torus orbit under μ is denoted by the symbol for the full polytope followed in parentheses by a list of conditions for Plücker coordinates that do or do not vanish on the orbit. A coordinate preceded by a backslash means that it vanishes, and a coordinate not preceded by a backslash means that it does not vanish. The vertices of the polytope denoted in this way are those whose preimages under μ satisfy all the listed conditions. To simplify notation, the Plücker coordinates π_i, π_i^*, and π_{ij} are denoted as i, i^*, and ij, respectively. The complement of a polytope is indicated by a superscript c.

Definition. For $k = 1, 2, 3, 4$, let $\mathcal{S}_k$ denote the $sl(3, \mathbf{C})$ subalgebra of $sl(4, \mathbf{C})$ generated by the three roots $L_i - L_j$ with $i, j \neq k$. The action of $\mathcal{S}_k$ on an irreducible representation Γ of $sl(4, \mathbf{C})$ decomposes it into a direct sum of irreducible representations γ_l of $sl(3, \mathbf{C})$. Every weight space of $\mathcal{S}_k$ in Γ with respect to the diagonal Cartan subalgebra is contained in a weight space for the action of the whole algebra $sl(4, \mathbf{C})$. It therefore corresponds to a weight in $\mathcal{H}_{\mathbf{R}}^*$. In particular, each weight space of $\mathcal{S}_k$ in a component γ_l corresponds to a

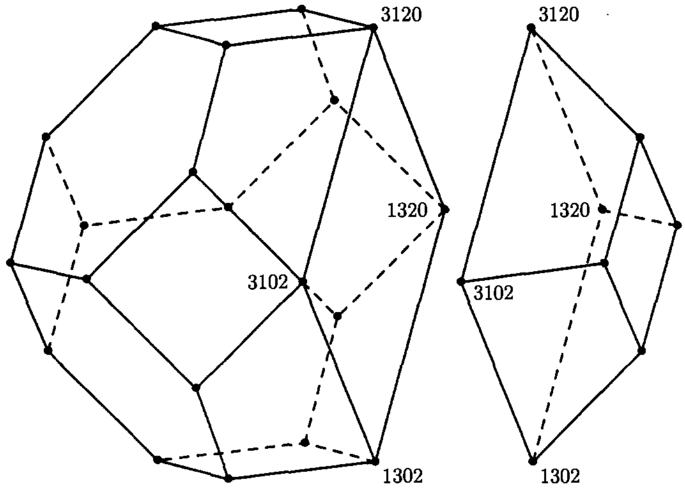


Fig. 7. A splitting of ◇ along an interior square.

weight in the weight polytope of Γ. We will refer to the convex hull of these weights as the *weight polytope* of γ_l in $\mathcal{H}^*_{\mathbf{R}}$.

The adjoint action of $\mathcal{S}_k$ on $sl(4, \mathbf{C})$ decomposes it into a direct sum of irreducible representations of $sl(3, \mathbf{C})$, one of which is isomorphic to the adjoint representation of $sl(3, \mathbf{C})$. The weight polytope in $\mathcal{H}^*_{\mathbf{R}}$ of this component is the interior hexagon $\triangle(\backslash k \backslash k^*)$, which splits $\triangle$ into the congruent polytopes $\triangle(\backslash k)$ and $\triangle(\backslash k^*)$.

Notation. Let $\Gamma_{a,b}$ denote the irreducible representation of $sl(3, \mathbf{C})$ with highest weight $aw_1 + bw_2$, where $a, b \in \mathbf{Z} \geq 0$ and w_1, w_2 are the highest weights of the standard representation and its dual, respectively.

The decomposition of $\Gamma_{G/B}$ into irreducible representations of $\mathcal{S}_k$ contains two copies of the adjoint representation of $sl(3, \mathbf{C})$. Their weight polytopes in $\mathcal{H}^*_{\mathbf{R}}$ are the two opposite hexagons $\diamond(k)$ and $\diamond(k^*)$. The decomposition also contains one copy of each of the representations $\Gamma_{2,1}$ and $\Gamma_{1,2}$. Their weight polytopes in $\mathcal{H}^*_{\mathbf{R}}$ are the two irregular hexagons parallel to $\diamond(k)$ and $\diamond(k^*)$ that split $\diamond$ into the pair of polytopes $\diamond(\backslash k)$ and $\diamond(\backslash k)^{\mathrm{c}}$ and the pair $\diamond(\backslash k^*)$ and $\diamond(\backslash k^*)^{\mathrm{c}}$, respectively.

Definition. For $ij = 12$, 13, and 14, denoted by $\mathcal{T}_{ij}$, the subalgebra of $sl(4, \mathbf{C})$ generated by the pair of roots $L_1 - L_2$ and $L_3 - L_4$, the pair $L_1 - L_3$ and $L_2 - L_4$, and the pair $L_1 - L_4$ and $L_2 - L_3$, respectively. Each $\mathcal{T}_{ij}$ is isomorphic to $sl(2, \mathbf{C}) \oplus sl(2, \mathbf{C})$. The action of $\mathcal{T}_{ij}$ on an irreducible representation of $sl(4, \mathbf{C})$ decomposes it into a direct sum of irreducible representations γ_k of $sl(2, \mathbf{C}) \oplus sl(2, \mathbf{C})$. Every weight space of $\mathcal{T}_{ij}$ in Γ with respect to the diagonal Cartan subalgebra is contained in a weight space for the action of $sl(4, \mathbf{C})$ and therefore corresponds to a weight in $\mathcal{H}^*_{\mathbf{R}}$. As before, we refer to the convex hull of these weights for a given component γ_k as the *weight polytope* of γ_k in $\mathcal{H}^*_{\mathbf{R}}$.

The action of $\mathcal{T}_{ij}$ on $\wedge^2 S$ has an irreducible component $\Gamma_1 \otimes \Gamma_1$, where Γ_1 is the standard representation of $sl(2,\mathbf{C})$. Its weight polytope in $\mathcal{H}^*_{\mathbf{R}}$ is a square that splits the octahedron into the congruent polytopes $\square(\backslash ij)$ and $\square(\backslash kl)$, where $\{i,j\} \cap \{k,l\} = \emptyset$. In the decomposition of $\Gamma_{G/B}$ under the action of $\mathcal{T}_{ij}$, there are two copies of $\Gamma_1 \otimes \Gamma_1$. Their weight polytopes in $\mathcal{H}^*_{\mathbf{R}}$ are the pair of opposite square faces $\diamond(ij)$ and $\diamond(kl)$. The decomposition also contains two copies of the representation $sl(2,\mathbf{C}) \otimes sl(2,\mathbf{C})$. Their weight polytopes in $\mathcal{H}^*_{\mathbf{R}}$ split $\diamond$ along two interior squares parallel to $\diamond(ij)$ and $\diamond(kl)$, producing the pair of polytopes $\diamond(\backslash ij)$ and $\diamond(\backslash ij)^{\mathrm{c}}$ and the pair $\diamond(\backslash kl)$ and $\diamond(\backslash kl)^{\mathrm{c}}$.

3. The Toda lattice of $sl(4,\mathbf{C})$

From here on, dimension refers to complex dimension, and G is $Sl(4,\mathbf{C})$. Here $\epsilon + \mathcal{B}_-$ is 9-dimensional, and a generic symplectic leaf is 8-dimensional. In addition to the three invariants $(1/k)\mathrm{tr}X^k$, $k = 2, 3, 4$, there are two 1-chop integrals: I_{11}, a Casimir, and I_{21}, a fourth constant of motion. I_{21} will sometimes be denoted simply as I.

Fix distinct eigenvalues λ_i and consider the 6-dimensional isospectral set $(\epsilon + \mathcal{B}_-)_\Lambda$. It contains a 2-parameter family of 4-dimensional level sets parametrized by I_{11} and I_{21} and is embedded in the flag manifold G/B by the torus embedding. In the flag manifold, a typical level set is essentially a 1-parameter family of 3-dimensional torus orbits; each orbit is generated by the diagonal $(\mathbf{C}^*)^3$ torus, which corresponds to the Hamiltonian flows of the integrals $(1/k)\mathrm{tr}X^k$. The flow of I_{21} evolves along the $\mathbf{P}^1$-fiber of the projection $\Pi_1 : G/B \to G/P_1$ and generates the 1-parameter family of orbits.

3.1. The general and special level sets

The elements $X \in \epsilon + \mathcal{B}_-$ have the form

$$X = \begin{pmatrix} f_1 & 1 & 0 & 0 \\ g_1 & f_2 & 1 & 0 \\ h_1 & g_2 & f_3 & 1 \\ k & h_2 & g_3 & f_4 \end{pmatrix},$$

with $\sum_{i=1}^4 f_i = 0$, and the 1-chop integrals I_{11} and I_{21} are the coefficients of the polynomial

$$\det \begin{pmatrix} \lambda - f_2 + (g_1 h_2)/k & -1 + (g_1 g_3)/k \\ -g_2 + (h_1 h_2)/k & \lambda - f_3 + (h_1 g_3)/k \end{pmatrix}.$$

It is found in [3] that in the flag manifold, these integrals have the expressions

$$I_{11} = \frac{\sum_{i=1}^4 \sigma_2(\hat{i})\pi_i\pi_i^*}{\sum_{i=1}^4 \sigma_1(\hat{i})\pi_i\pi_i^*},$$

and

$$I_{21} = \frac{\sum_{i=1}^{4} \sigma_3(\hat{i})\pi_i\pi_i^*}{\sum_{i=1}^{4} \sigma_1(\hat{i})\pi_i\pi_i^*},$$

where $\sigma_k(\hat{i})$ denotes the kth symmetric polynomial in the three eigenvalues different from λ_i.

Let LSV_I denote a level set variety of the 1-chop integrals in G/B, defined by the equations

$$I_{11}\left(\sum \sigma_1(\hat{i})\pi_i\pi_i^*\right) - \sum \sigma_2(\hat{i})\pi_i\pi_i^* = 0, \tag{6}$$

and

$$I_{21}\left(\sum \sigma_1(\hat{i})\pi_i\pi_i^*\right) - \sum \sigma_3(\hat{i})\pi_i\pi_i^* = 0, \tag{7}$$

for fixed values of $I_{11} \in \mathbf{C}$ and $I_{21} \in \mathbf{P}^1$, where (7) becomes $\sum \sigma_1(\hat{i})\pi_i\pi_i^* = 0$ when $I_{21} = \infty$. Let λ_{11} and λ_{21} denote the eigenvalues of the 2×2 matrix ϕ_1 in Proposition 2.2.

Definition. LSV_I will be called a *general level set* if neither λ_{11} nor λ_{21} coincides with an eigenvalue λ_i. If there is exactly one coincidence of eigenvalues $\lambda_{r1} = \lambda_i$, then LSV_I will be called a *special level set of type I*, and if λ_{11} and λ_{21} are both eigenvalues of $X \in (\epsilon + \mathcal{B}_-)_\Lambda$, then LSV_I will be called a *special level set of type II*.

3.2. *Geometry of the level sets in G/P_1*

Since (6) and (7) involve only the Plücker coordinates on the manifold of partial flags G/P_1, we consider the image of a level set LSV_I under the projection $\Pi_1 : G/B \to G/P_1$. In [3] it is found that if LSV_I is general with $I_{12} \neq \infty$, then $\Pi_1(\mathrm{LSV}_I)$ is the closure of a single generic torus orbit; a similar argument also works when $I_{21} = \infty$. This orbit is generated by the flows of the integrals $(1/k)\mathrm{tr}X^k$, and its image under the moment map is the full polytope, Δ.

The special level sets are characterized by splittings of Δ along its interior hexagons.

Proposition 3.1.

1. *Let LSV_I be a special level set of type I for which λ_i is an eigenvalue of ϕ_1. Then $\Pi_1(LSV_I)$ is the union of the closures of two nongeneric torus orbits, $\mathcal{O}^i$ and $\mathcal{O}^{i^*}$, on which π_i and π_i^*, respectively, vanish. Their polytopes, $\Delta(\backslash i)$ and $\Delta(\backslash i^*)$, are obtained by splitting Δ along the interior hexagon $\Delta(\backslash i \backslash i^*)$. The closures of $\mathcal{O}^i$ and $\mathcal{O}^{i^*}$ intersect along a common 2-dimensional orbit, $\mathcal{O}^{ii^*}$, whose polytope is this interior hexagon.*
2. *If LSV_I is a special set of type II for which λ_i and λ_j are eigenvalues of ϕ_1, then $\Pi_1(LSV_I)$ is the union of the closures of the four (unique) torus orbits whose polytopes are obtained by splitting Δ simultaneously along the interior hexagons $\Delta(\backslash i \backslash i^*)$ and $\Delta(\backslash j \backslash j^*)$.*

Proof. (1) By the proof of Lemma 4.4 in [3], the coincidence of an eigenvalue of ϕ_1 with λ_i is equivalent to $\pi_i \pi_i^* = 0$. So on each 3-dimensional torus orbit in $\Pi_1(\mathrm{LSV}_I)$, exactly one of π_i and π_i^* vanishes. The proof of Theorem 5.1 in [3] shows that the degree of $\Pi_1(\mathrm{LSV}_I)$ is the same as the degree of a generic torus orbit in G/P_1, and by Proposition 2.10 in [8], the degree of a 3-dimensional orbit in G/P_1 is proportional to the volume of its image polytope under the moment map. The polytopes of the two strata where either π_i or π_i^* is the only vanishing Plücker coordinate are obtained by splitting $\triangle$ along the interior hexagon $\triangle(\backslash i \backslash i^*)$. Since each of these polytopes has half the volume of the full polytope and the degree of $\Pi_1(\mathrm{LSV}_I)$ is the same as the degree of a generic orbit, $\Pi_1(\mathrm{LSV}_I)$ contains exactly two 3-dimensional orbits (on each of which exactly one of π_i and π_i^* vanishes). That $\Pi_1(\mathrm{LSV}_I)$ contains one orbit of each type follows from the symmetry of (6) and (7) in π_k and π_k^*.

Part (2) follows from a similar argument and the fact that there is a unique torus orbit in the stratum associated to each of the four polytopes obtained by splitting $\triangle$ simultaneously along the interior hexagons $\triangle(\backslash i \backslash i^*)$ and $\triangle(\backslash j \backslash j^*)$. □

3.3. *Geometry of the level sets in G/B*

As I_{11} and I_{21} vary, (6) and (7) cut out a 2-parameter family of level set varieties in the flag manifold. If $X \in G/P_1$ has trivial stabilizer, then each point in $\Pi_1^{-1}(X)$ lies in a different torus orbit in G/B and also has trivial stabilizer. So each 3-dimensional orbit in $\Pi_1(\mathrm{LSV}_I)$ lifts to a family of 3-dimensional orbits in G/B parametrized by $\mathbf{P}^1$. A general level set has the form

$$\mathrm{LSV}_I = \Pi_1^{-1}(\mathcal{O}) \cup \mathcal{Z}_I, \tag{8}$$

where $\mathcal{O}$ is a generic torus orbit in G/P_1 and $\mathcal{Z}_I$ the common intersection of the varieties LSV_I. $\Pi_1^{-1}(\mathcal{O})$ is a 1-parameter family of torus orbits in G/B, almost all of which are generic.

A special level set of type I has the form

$$\mathrm{LSV}_I = \Pi_1^{-1}(\mathcal{O}^i) \cup \Pi_1^{-1}(\mathcal{O}^{i^*}) \cup \Pi_1^{-1}(\mathcal{O}^{ii^*}) \cup \mathcal{Z}_I. \tag{9}$$

Each of $\Pi_1^{-1}(\mathcal{O}^i)$ and $\Pi_1^{-1}(\mathcal{O}^{i^*})$ is a 1-parameter family of nongeneric orbits, almost all of which belong to the stratum with polytope $\diamond(\backslash i)$, respectively $\diamond(\backslash i^*)$. $\Pi_1^{-1}(\mathcal{O}^{ii^*})$ contains a unique 3-dimensional orbit, whose polytope is $\diamond(\backslash i \backslash i^*)$.

3.3.1. *The base locus $\mathcal{Z}_I$*

Throughout the paper, we refer to the common intersection of a set of varieties as their *base locus*. The base locus of the varieties $\Pi_1(\mathrm{LSV}_I)$ in G/P_1 is the union of the torus orbits that correspond under the moment map to a face, edge, or vertex of $\triangle$. Each of these orbits is unique in its stratum. The base locus $\mathcal{Z}_I$ of the varieties LSV_I in G/B is the inverse image under Π_1 of this union. $\mathcal{Z}_I$ consists of the strata in the flag manifold that correspond to the faces, edges, and vertices of $\diamond$, together with the closures of the six unique 3-dimensional orbits $\mathcal{Q}_{\diamond(\backslash ij)^{\mathrm{c}}}$, whose polytopes are $\diamond(\backslash ij)^{\mathrm{c}}$ (Fig. 7).

3.3.2. The cross-ratio

The cross-ratio is a torus-invariant function defined on an open dense subset of the flag manifold. It depends only on $Gr(2,4)$:

$$c = \frac{\pi_{12}\pi_{34}}{\pi_{13}\pi_{24}}. \tag{10}$$

In Section 4.1, we will see that another constant of motion J, derived from $so(6,\mathbf{C})$, is a linear fractional transformation of c. Here we show that the cross-ratio is a two-to-one branched map on the set of 3-dimensional orbits in a general level set of I_{21}.

Let $\mathcal{Z}_c$ denote the base locus of the varieties in G/B defined by the polynomials

$$\pi_{12}\pi_{34} - c\pi_{13}\pi_{24} = 0,$$

as c varies over $\mathbf{C}$. $\mathcal{Z}_c$ is the union of the strata on which three or more $\pi_{\alpha\beta}$ vanish.

Proposition 3.2. *Consider a general level set $LSV_I = \Pi_1^{-1}(\mathcal{O}) \cup \mathcal{Z}_I$, and let $H\backslash\Pi_1^{-1}(\mathcal{O})$ denote the quotient of $\Pi_1^{-1}(\mathcal{O})$ by the torus action.*

1. *$\Pi_1^{-1}(\mathcal{O})$ is contained in $(G/B) - \mathcal{Z}_c$, and $c : H\backslash\Pi_1^{-1}(\mathcal{O}) = \mathbf{P}^1 \to \mathbf{P}^1$ is a two-to-one map branched over two distinct points.*
2. *Every torus orbit in $\Pi_1^{-1}(\mathcal{O})$ whose cross-ratio is not 0, 1, or ∞ is generic. If $c \in \{0,1,\infty\}$ and c is not a branch point, then the two orbits with cross-ratio c belong to the nongeneric strata with polytopes $\diamond(\backslash ij)$ and $\diamond(\backslash kl)$, where $\{i,j\}\cup\{k,l\} = \{1,2,3,4\}$, and $(i,j) = (1,2), (1,4)$, or $(1,3)$ when $c = 0, 1$, or ∞, respectively. If $c \in \{0,1,\infty\}$ and c is a branch point, then the unique torus orbit with cross-ratio c belongs to the stratum with polytope $\diamond(\backslash ij\backslash kl)$.*

Proof. Choose a point X mod P_1 in $\mathcal{O}$, and let

$$X(u) = X \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & u & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{when } u \in \mathbf{C},$$

and

$$X \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{when } u = \infty.$$

Then $X(u)$ mod B for $u \in \mathbf{P}^1$ is a section of the torus orbits in $\Pi_1^{-1}(\mathcal{O})$, and

$$c(u) = \frac{\pi_{12}(u)\pi_{34}(u)}{\pi_{13}(u)\pi_{24}(u)}$$

expresses the cross-ratio on $H\backslash\Pi_1^{-1}(\mathcal{O})$ as a function of u. Each $\pi_{ij}(u)$ is either of degree 1 in u or constant.

In order for an orbit to be in $\mathcal{Z}_c$, three or more $\pi_{\alpha\beta}$ must vanish. The relations among the Plücker coordinates that define G/B as a variety in $P(S) \times P(\wedge^2 S) \times P(S^*)$ imply that in this case, some π_i or π_i^* must also vanish. But $\mathcal{O}$ is generic so $\Pi_1^{-1}(\mathcal{O}) \cap \mathcal{Z}_c$ must be empty. One may check that since no π_i or π_i^* vanishes, no $\pi_{\alpha\beta}(u)$ vanishes identically and at most one factor in $c(u)$ is constant. Also, if exactly two $\pi_{\alpha\beta}$ vanish on the same orbit, then they must be π_{ij} and π_{kl}, where $\{k, l\} \cap \{i, j\} = \emptyset$. So no linear factor in the numerator of $c(u)$ vanishes at the same value of u as a linear factor in the denominator. This means that no two linear factors in the expression of c cancel so that $c : \mathbf{P}^1 \to \mathbf{P}^1$ is a meromorphic function of degree two. By the Riemann–Hurwitz formula, it is branched over two distinct points.

Part (2) is verified by observing which coordinates vanish when $c = 0, 1$, or ∞. □

It will be shown that the torus orbits in $H\backslash\Pi_1^{-1}(\mathcal{O})$ with $c = 0, 1$, or ∞ mark the intersections of LSV_I with the special level sets of J.

Proposition 3.3. *Let* LSV_I *be a special level set of type I as in (9)*

1. *Each of* $\Pi_1^{-1}(\mathcal{O}^{i^*})$ *and* $\Pi_1^{-1}(\mathcal{O}^{i^*})$ *contains a unique orbit in the base locus* $\mathcal{Z}_c$ *with polytope* $\diamond(\backslash i^*)^c$, *respectively* $\diamond(\backslash i)^c$. *If the cross-ratio is defined on each of these orbits by its limit in* $\Pi_1^{-1}(\mathcal{O}^i)$, *respectively* $\Pi_1^{-1}(\mathcal{O}^{i^*})$, *then the two limits coincide, and the mappings* $c : H\backslash\Pi_1^{-1}(\mathcal{O}^i) = \mathbf{P}^1 \to \mathbf{P}^1$ *and* $c : H\backslash\Pi_1^{-1}(\mathcal{O}^{i^*}) = \mathbf{P}^1 \to \mathbf{P}^1$ *are both bijective.*
2. $\Pi_1^{-1}(\mathcal{O}^{ii^*})$ *contains a unique 3-dimensional orbit,* $\tilde{\mathcal{O}}^{ii^*}$. *The value of c on* $\tilde{\mathcal{O}}^{ii^*}$ *is the same as its extended value on the two orbits in part 1, and this value is not* 0, 1, or ∞.
3. *All but four of the torus orbits in* $\Pi_1^{-1}(\mathcal{O}^i)$, *respectively* $\Pi_1^{-1}(\mathcal{O}^{i^*})$, *belong to the stratum with polytope* $\diamond(\backslash i)$, *respectively* $\diamond(\backslash i^*)$. *In each case, the remaining four orbits are those with cross-ratios 0, 1, and* ∞ *and the orbit belonging to* $\mathcal{Z}_c$.

Proof. Let X mod P_1 be in $\mathcal{O}^i$, and let $X(u)$ and $c(u)$ be as in the proof of Proposition 3.2. As before, since π_i is the only vanishing Plücker coordinate on $\mathcal{O}^i$, no $\pi_{\alpha\beta}(u)$ vanishes identically and at most one factor in $c(u)$ can be constant. From the relations among the Plücker coordinates that define the image of G/B in $P(S) \times P(\wedge^2 S) \times P(S^*)$, $\pi_i = 0$ implies that $\pi_{ij}(u)$ is proportional to $\pi_{ik}(u)$ for j, k distinct and not equal to i. So all three $\pi_{i\gamma}(u)$ vanish on the same torus orbit, and two linear factors cancel in the expression of c. A second cancelation is not possible so $c(u)$ has degree 1. The single value of u for which all three $\pi_{i\gamma}(u)$ vanish corresponds to an orbit with polytope $\diamond(\backslash i^*)^c$, on which the cross-ratio is undefined. However, after canceling the proportional factors in $c(u)$, the result gives a well-defined value on this orbit, and we have a 1:1 mapping $c : H\backslash\Pi_1^{-1}(\mathcal{O}^i) = \mathbf{P}^1 \to \mathbf{P}^1$. The statement for $\Pi_1^{-1}(\mathcal{O}^{i^*})$ is shown similarly.

For part (2), take $i = 1$ to fix notation. A point in $\mathcal{O}^{11^*}$ has the form $[0 : 1 : p : q] \times [0 : 1 : r : s]$ with $1 + pr + qs = 0$ and $pqrs \neq 0$, and a point in $\Pi_1^{-1}(\mathcal{O}^{11^*})$ with trivial stabilizer can be written $[0 : 1 : p : q] \times [z : pz : qz : s : -r : 1] \times [0 : 1 : r : s]$ with

$z \neq 0$. One checks that any two such points lie in the same torus orbit and that the value of c on this orbit is as stated.

Part (3) follows from observing which coordinates vanish on the orbits in $\Pi_1^{-1}(\mathcal{O}^i)$ and $\Pi_1^{-1}(\mathcal{O}^{i*})$. □

3.3.3. The generic leaf with Casimir zero

From here through Section 6, we fix the symplectic leaf $I_{11} = 0$ for simplicity of the exposition. The geometry described for this leaf extends to all the generic leaves through the generalization in Section 7.

Let $\mathcal{V}$ denote the subvariety of G/B defined by (6) with $I_{11} = 0$, and consider the family of level sets in $\mathcal{V}$ cut out by (7) for $I_{21} \in \mathbf{C}$ and by $\sum \sigma_1(\hat{i})\pi_i\pi_i^* = 0$ when $I_{21} = \infty$. Since $I_{11} = 0$, the characteristic polynomial of $\phi_1(X)$ is $\lambda^2 + I_{21} = 0$. So λ_i is an eigenvalue of $\phi_1(X)$ (the level set is special) exactly when $I_{21} = -\lambda_i^2$.

The type and number of special level sets depend on whether s_3 vanishes in the characteristic polynomial $\lambda^4 - s_2\lambda^2 - s_3\lambda - s_4$. If $s_3 = 0$, the spectrum has the form $\{\lambda_{i_1}, -\lambda_{i_1}, \lambda_{i_2}, -\lambda_{i_2}\}$. The values $-\lambda_i^2$ coincide in two pairs, and there are two special level sets of type II. If $s_3 \neq 0$, then $\lambda_j \neq -\lambda_i$ for $i \neq j$. In this case the $-\lambda_i^2$ are distinct, and $\mathcal{V}$ contains four special level sets of type I. Recall that there are four splittings of Δ along interior hexagons. If $s_3 \neq 0$, they correspond to the four special level sets in $\mathcal{V}$, and if $s_3 = 0$, the splittings occur in two pairs, yielding two special level sets of type II. Throughout the rest of the paper, we assume that $S_3 \neq 0$.

4. The Toda lattice of *so*(6, *C*)

4.1. The integral J

Recall that under the torus embedding, the integral I_{21}, obtained by a chopping construction on $sl(4, \mathbf{C})$, is defined in terms of the partial flag manifold $\{V^1 \subset V^3 \subset \mathbf{C}^4\}$. A similar chopping construction on the isomorphic algebra $so(6, \mathbf{C})$ yields another constant of motion, J. It turns out that this integral, is defined purely in terms of the complementary partial flag manifold, $\{V^2 \subset \mathbf{C}^4\}$. The fibration of $\mathcal{V}$ by the level sets of J is transverse to and has a similar structure to the fibration of $\mathcal{V}$ by the level sets of I_{21}.

Consider the isomorphism $\rho : sl(4, \mathbf{C}) \to so(6, \mathbf{C})$ induced by the representation $\wedge^2\mathbf{C}^4$ of $sl(4, \mathbf{C})$. For $X \in sl(4, \mathbf{C})$, let $\rho(X)$ be written with 0's along the antidiagonal, and denote by $(\rho(X) - \lambda I)_{(1)}$ the "1-chop" of the matrix $\rho(X) - \lambda I$, that is, the result of removing its first row and last column. Then $\det(\rho(X) - \lambda I)_{(1)}$ is a polynomial of the form $\alpha\lambda^3 + \beta\lambda$, and β/α is a constant of motion, J [3].

For $X \in (\epsilon + \mathcal{B}_-)_\Lambda$, the eigenvalues of $\rho(X)$ are $\{\lambda_i + \lambda_j\}_{i \neq j}$. Since $\sum_{i=1}^4 \lambda_i = 0$, they satisfy $\lambda_k + \lambda_l = -(\lambda_i + \lambda_j)$ when $\{i, j\} \cap \{k, l\} = \emptyset$. The roots of the polynomial $\det(\rho(X) - \lambda I)_{(1)}$ are zero and $\pm\gamma$, where $\gamma^2 = -J$. So if $s_3 \neq 0$ (so that no $\lambda_i + \lambda_j$ is zero), then a root of this polynomial coincides with an eigenvalue of $\rho(X)$ precisely when $J = -(\lambda_i + \lambda_j)^2$.

Under the torus embedding of $(\epsilon + \mathcal{B}_-)_\Lambda$ into the flag manifold, J is a linear fractional transformation of the cross-ratio:

$$J = \frac{A_\Lambda A'_\Lambda c - B_\Lambda B'_\Lambda}{A_\Lambda c - B_\Lambda}, \tag{11}$$

where

$$A_\Lambda = (\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4),$$

$$A'_\Lambda = (\lambda_1 + \lambda_3)(\lambda_2 + \lambda_4),$$

$$B_\Lambda = (\lambda_1 - \lambda_2)(\lambda_3 - \lambda_4),$$

$$B'_\Lambda = (\lambda_1 + \lambda_2)(\lambda_3 + \lambda_4).$$

This formula holds on an arbitrary generic leaf. It was found through calculations using MAPLE and the matrices $X(I_{11}, I_{21}, u)$ in Section 7. The three values $J = -(\lambda_i + \lambda_j)^2$ correspond to $c = 0$, 1, and ∞.

Let LSV_J denote a subvariety of $\mathcal{V}$ defined by

$$(A_\Lambda A'_\Lambda c - B_\Lambda B'_\Lambda) - J(A_\Lambda c - B_\Lambda) = 0 \tag{12}$$

when $J \in \mathbf{C}$ and by $A_\Lambda c - B_\Lambda = 0$ when $J = \infty$. LSV_J will be called a *general level set* if $J \neq -(\lambda_i + \lambda_j)^2$ and a *special level set* if $J = -(\lambda_i + \lambda_j)^2$.

4.2. *The level sets of J in Gr(2,4)*

The geometry of the varieties $\Pi_2(\mathrm{LSV}_J)$ in $Gr(2,4)$ is analogous to the geometry of the varieties $\Pi_1(\mathrm{LSV}_I)$ in G/P_1. The general level sets correspond to generic torus orbits, while the special level sets are characterized by splittings of the moment polytope.

Proposition 4.1.

1. *If LSV_J is a general level set, then $\Pi_2(LSV_J)$ is the closure of a single generic torus orbit in Gr(2,4).*
2. *Let LSV_J be the special level set with $J = -(\lambda_i + \lambda_j)^2$. Then $\Pi_2(LSV_J)$ is the union of the closures of two nongeneric torus orbits, $\mathcal{Q}^{ij}$ and $\mathcal{Q}^{kl}$, on which π_{ij} and π_{kl}, respectively, vanish, where $\{i, j\} \cap \{k, l\} = \emptyset$. Their polytopes, $\square(\backslash ij)$ and $\square(\backslash kl)$, are obtained by splitting the octahedron along an interior square. The two orbits are unique in their strata, and they intersect along a 2-dimensional orbit, $\mathcal{Q}^{(ij)(kl)}$, that corresponds under the moment map to the interior square.*

Proof. Since J is a linear fractional transformation of c, a level set of J is a level set of c. The cross-ratio of an element

$$\begin{pmatrix} z_1 & w_1 \\ z_2 & w_2 \\ z_3 & w_3 \\ z_4 & w_4 \end{pmatrix} \bmod Gl(2, \mathbf{C})$$

in $Gr(2,4)$ is the cross-ratio of the four points $p_i = [z_i : w_i]$ in $\mathbf{P}^1$, that is, the image of the point $[z_1 : w_1]$ under the linear fractional transformation that takes p_2 to 0, p_3 to ∞, and p_4 to 1. Consider the collection $C^0_{4,2}$ of ordered 4-tuples of nonzero points (z_i, w_i) in $\mathbf{C}^2$. The torus H acts on $C^0_{4,2}$ on the left by sending (z_i, w_i) to $(h_i z_i, h_i w_i)$; $H\backslash C^0_{4,2}$ is the collection of ordered sets of four points in $\mathbf{P}^1$. On the other hand, $Gl(2, \mathbf{C})$ acts on $C^0_{4,2}$ on the right, and the quotient $C^0_{4,2}\backslash Gl(2, \mathbf{C})$ is an open dense subset of $Gr(2, 4)$. The quotient of $C^0_{4,2}$ by both H and $Gl(2, \mathbf{C})$ is open and dense in the space of torus orbits in $Gr(2,4)$ and contains in particular all the 3-dimensional orbits.

If $X \bmod Gl(2, \mathbf{C})$ and $Y \bmod Gl(2, \mathbf{C})$ have the same value of $c \neq 0, 1, \infty$, then there is a linear fractional transformation that takes the ordered set of four points in $\mathbf{P}^1$ determined by $H \cdot X$ into the one determined by $H \cdot Y$. So there is one torus orbit with this value of c, and it is generic since no $\pi_{\alpha\beta}$ vanishes. If $J = -(\lambda_i + \lambda_j)^2$, then c is 0, 1, or ∞, and it is easy to check that the level set consists in the closures of two nongeneric orbits as stated. □

4.3. *The level sets of J in $\mathcal{V}$*

As in the case of I_{21}, the geometry of the level sets of J in $\mathcal{V}$ depends on whether s_3 vanishes in the characteristic polynomial. If $s_3 \neq 0$, then the three special level sets $J = -(\lambda_i + \lambda_j)^2$ are of the same type. If $s_3 = 0$, then the spectrum has the form $\Lambda = \{\lambda_1, -\lambda_1, \lambda_2, -\lambda_2\}$; the level sets with $J = -(\lambda_1 + \lambda_2)^2$ and $J = -(\lambda_1 - \lambda_2)^2$ are of the same type, and the level set with $J = 0$ has a different structure. Here we focus on the case where no special value of J is zero (cf. Proposition 5.4); we continue to assume that $S_3 \neq 0$.

The structure of the J-fibration of $\mathcal{V}$ is parallel to that of the I_{21}-fibration. A general level set has the form

$$\mathrm{LSV}_J = \Pi_2|_{\mathcal{V}}^{-1}(\mathcal{Q}) \cup \mathcal{Z}_J, \tag{13}$$

where $\mathcal{Q}$ is a generic torus orbit in $Gr(2,4)$ and $\mathcal{Z}_J$ is the base locus of the varieties in (12). As in (8), $\Pi_2|_{\mathcal{V}}^{-1}(\mathcal{Q})$ is a 1-parameter family of torus orbits in G/B, almost all of which are generic. A special level set is a union

$$\mathrm{LSV}_J = \Pi_2|_{\mathcal{V}}^{-1}(\mathcal{Q}^{ij}) \cup \Pi_2|_{\mathcal{V}}^{-1}(\mathcal{Q}^{kl}) \cup \Pi_2|_{\mathcal{V}}^{-1}(\mathcal{Q}^{(ij)(kl)}) \cup \mathcal{Z}_J \tag{14}$$

(cf. Proposition 4.1). As in (9), $\Pi_2|_{\mathcal{V}}^{-1}(\mathcal{Q}^{ij})$ and $\Pi_2|_{\mathcal{V}}^{-1}(\mathcal{Q}^{kl})$ are 1-parameter families of nongeneric orbits, and $\Pi_2|_{\mathcal{V}}^{-1}(\mathcal{Q}^{(ij)(kl)})$ contains a unique 3-dimensional orbit.

4.3.1. *The base locus $\mathcal{Z}_J$*

Consider the common intersection, $\mathcal{Z}_J$, of the varieties (12). Since the level sets of J in $\mathcal{V}$ coincide with the level sets of the cross-ratio, $\mathcal{Z}_J$ is the intersection of $\mathcal{Z}_c$ with $\mathcal{V}$. In particular, it contains all orbits whose images under $\mu_{G/B}$ lie in the boundary of the moment polytope. $\mathcal{Z}_J$ also contains the closures of eight 3-dimensional orbits, $\mathcal{O}_{\diamond(\backslash i^*)^c}$ and $\mathcal{O}_{\diamond(\backslash i)^c}$ for $i = 1, 2, 3, 4$, whose polytopes are $\diamond(\backslash i^*)^c$ and $\diamond(\backslash i)^c$, respectively. (These are the orbits in $\mathcal{Z}_c$ contained in the special level sets LSV_J in $\mathcal{V}$ as in Proposition 3.3.)

4.3.2. The 1-parameter families of torus orbits

The nongeneric 3-dimensional orbits in a general level set of J mark where the level set intersects the special level sets of I_{21}. Typically, there are two such intersections for each special value of I_{21}.

Proposition 4.2. *Let LSV_J be a general level set as in (13) with $s_3 \neq 0$. Then $(\Pi_2|_{\mathcal{V}})^{-1}(\mathcal{Q}) \cap \mathcal{Z}_I$ is empty, and each orbit with $I_{21} \neq -\lambda_i^2$ is generic. If $J \neq s_3/(2\lambda_i)$, there are two non-generic orbits in $(\Pi_2|_{\mathcal{V}})^{-1}(\mathcal{Q})$ with $I_{21} = -\lambda_i^2$, and if $J = s_3/(2\lambda_i)$, there is a unique such orbit.*

It will be useful to take a section of the 3-dimensional orbits in each general level set of I_{21} in $\mathcal{V}$. Consider the matrices $X(I_{21}, u)$ in $(\epsilon + \mathcal{B}_-)_\Lambda$, parametrized by I_{21} and u, where I_{11} is identically zero.

$$X(I_{21}, u) = Z(u) L C_\Lambda L^{-1} Z^{-1}(u)$$
$$= \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & u & 1 & 0 \\ 0 & -(I_{21} + u^2) & -u & 1 \\ s_4 - s_3 + s_2 - 1 & u(s_2 - 1 + I_{21}) + s_3 - s_2 + 1 - I_{21} & s_2 - 1 + I_{21} & 1 \end{pmatrix}, \tag{15}$$

where

$$L^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ -1 & -I_{21} & -1 & 1 \end{pmatrix}, \qquad Z^{-1}(u) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & u-1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The action of $Z(u)$ generates the flow of I_{21}. Under the torus embedding, u extends to $\mathbf{P}^1$ and parametrizes the set of torus orbits in $\Pi_1^{-1}(\mathcal{O})$ for each fixed $I_{21} \neq -\lambda_i^2$.

Evaluating J on $X(I_{21}, u)$ yields

$$J = -\frac{I_{21}^2 + s_2 I_{21} - s_3 u - s_4}{I_{21} + u^2}, \tag{16}$$

which gives a quadratic polynomial in u:

$$J u^2 - s_3 u + (I_{21}^2 + (s_2 + J) I_{21} - s_4) = 0. \tag{17}$$

Lemma 4.1. *In the special level set LSV_I with $I_{21} = -\lambda_i^2$, J takes the value $s_3/2\lambda_i$ on the orbit $\tilde{\mathcal{O}}^{ii^*}$.*

Proof. For $I_{21} = -\lambda_i^2$, $X(I_{21}, u)$ is a section of the torus orbits in the component $\Pi_1^{-1}(\mathcal{O}^{i^*})$. Since $J|_{\mathcal{V}}$ is equivalent to c, Proposition 3.3 implies that J is a one-to-one function from the set of orbits in $\Pi_1^{-1}(\mathcal{O}^{i^*})$ to $\mathbf{P}^1$. For $I_{21} = -\lambda_i^2$, the numerator in (16) is $\lambda_i^4 - s_2\lambda_i^2 - s_3 u - s_4$, which is equal to $s_3(\lambda_i - u)$ since λ_i is a root of the characteristic polynomial; so (16) reduces to $J = s_3/(\lambda_i + u)$. When $u = \lambda_i$, three $\pi_{\alpha\beta}$ vanish; this value of u corresponds to the unique orbit in $\Pi_1^{-1}(\mathcal{O}^{i^*})$ contained in the base locus $\mathcal{Z}_c$. By Proposition 3.3(b), the limiting value of c on this orbit is the same as its value on $\tilde{\mathcal{O}}^{ii^*}$. Since $u = \lambda_i$, this corresponds to $J = s_3/2\lambda_i$. □

Proof of Proposition 4.2. Observe first that none of the six 3-dimensional orbits in $\mathcal{Z}_I$ is contained in LSV_J since the values of c on these orbits are 0,1, and ∞, which correspond to the special values of J. If $I_{21} \neq -\lambda_i^2$, then the orbit belongs to a general level set LSV_I and is generic by Proposition 3.2(2).

The orbits with $I_{21} = -\lambda_i^2$ belong to a special level set LSV_I. By Proposition 3.3 and Lemma 4.1, if $J \neq s_3/2\lambda_i$, then there is one orbit in each of the components $\Pi_1^{-1}(\mathcal{O}^i)$ and $\Pi_1^{-1}(\mathcal{O}^{i^*})$ with this value of J. The only orbit in the level set $I_{21} = -\lambda_i^2$ with $J = s_3/(2\lambda_i)$ is $\tilde{\mathcal{O}}^{ii^*}$. □

The structure of a special level set of J in $\mathcal{V}$ is parallel to that of a special level set of I_{21} as described in Proposition 3.3.

Proposition 4.3. *Let LSV_J be the special level set with $J = -(\lambda_i + \lambda_j)^2$ as in* (14) *with* $s_3 \neq 0$.

1. $(\Pi_2|_{\mathcal{V}})^{-1}(\mathcal{Q}^{ij})$ *contains a unique orbit, $\mathcal{Q}_{\diamond(\backslash kl)^c}$, in the base locus $\mathcal{Z}_I$; its polytope is $\diamond(\backslash kl)^c$. If the function I_{21} is extended to this orbit to make it continuous on the quotient $H\backslash(\Pi_2|_{\mathcal{V}})^{-1}(\mathcal{Q}^{ij})$, then $H\backslash(\Pi_2|_{\mathcal{V}})^{-1}(\mathcal{Q}^{ij})$ is isomorphic to $\mathbf{P}^1$ and is parametrized by I_{21}.*
2. $\Pi_2|_{\mathcal{V}}^{-1}(\mathcal{Q}^{(ij)(kl)})$ *contains a unique 3-dimensional orbit, $\tilde{\mathcal{Q}}^{(ij)(kl)}$. Its value of I_{21} is the extended value of I_{21} on $\mathcal{Q}_{\diamond(\backslash kl)^c}$.*

Proof. Any 3-dimensional torus orbit in $\mathcal{V}$ that is sent to $\mathcal{Q}^{(ij)(kl)}$ by Π_2 satisfies $\pi_{ij} = \pi_{kl} = 0$ and has no other vanishing $\pi_{\alpha\beta}$. There is no such orbit in any special level set LSV_I, and such an orbit occurs in a general level set of I_{21} if and only if $J : H\backslash\Pi_1^{-1}(\mathcal{O}) \to \mathbf{P}^1$ is branched over $J = -(\lambda_i + \lambda_j)^2$ (see (11) and Proposition 3.2). For $I_{21} \neq -\lambda_i^2$, the two values of J over which this map is branched are those for which the two solutions u of (17) coincide. This happens where the discriminant,

$$D(I_{21}, J) = s_3^2 - 4J(I_{21}^2 + (s_2 + J)I_{21} - s_4), \tag{18}$$

of (17) as a polynomial in u, vanishes. Since this is quadratic in I_{21}, a given value of J typically occurs as a branch point for two distinct values of I_{21}. The values of J that occur only once as a branch point are those where the discriminant of (18) as a polynomial in I_{21} vanishes. This discriminant is

$$D'(J) = 16J(J^3 + 2s_2J^2 + (s_2^2 + 4s_4)J - s_3^2), \tag{19}$$

and its roots are $J = 0,\ -(\lambda_1 + \lambda_2)^2,\ -(\lambda_1 + \lambda_3)^2$, and $-(\lambda_1 + \lambda_4)^2$. $J = -(\lambda_i + \lambda_j)^2$ is therefore a branch point of $J : H\backslash\Pi_1^{-1}(\mathcal{O}) \to \mathbf{P}^1$ for only one value of I_{21}. Denote this value by a. The branching occurs at a unique torus orbit in the level set $I_{21} = a$, and its image under Π_2 is $\mathcal{Q}^{(ij)(kl)}$; denote this orbit by $\tilde{\mathcal{Q}}^{(ij)(kl)}$.

To be in $(\Pi_2|_{\mathcal{V}})^{-1}(\mathcal{Q}^{ij})$, π_{ij} must be the only $\pi_{\alpha\beta}$ that vanishes. For a general value of I_{21}, LSV_I contains a unique such orbit unless J is branched over $J = -(\lambda_i + \lambda_j)^2$.

In this case, $I_{21} = a$, and π_{ij} and π_{kl} vanish on the same orbit (Proposition 3.2(2)). In a special level set of I_{21}, only one of the components $\Pi_1^{-1}(\mathcal{O}^\gamma)$ and $\Pi_1^{-1}(\mathcal{O}^{\gamma^*})$ contains an orbit on which π_{ij} is the only vanishing $\pi_{\alpha\beta}$, and this orbit is unique (Proposition 3.3). Finally, the base locus $\mathcal{Z}_I$ also contains a unique such orbit, $\mathcal{Q}_{\diamond(\backslash kl)^c}$. I_{21} is therefore 1:1 on $(\Pi_2|_{\mathcal{V}})^{-1}(\mathcal{Q}^{ij}) - \mathcal{Q}_{\diamond(\backslash kl)^c}$ with image $\mathbf{P}^1 - \{a\}$. Extending it by continuity gives the value $I_{21} = a$ on $\mathcal{Q}_{\diamond(\backslash kl)^c}$. □

5. Monodromy

We now focus on the geometry of the level set varieties in the open subset of $\mathcal{V}$ obtained by removing the base loci $\mathcal{Z}_I$ and $\mathcal{Z}_J$. (The orbits in these base loci correspond to isospectral sets in lower-dimensional symplectic leaves of $\epsilon + \mathcal{B}_-$ [11].)

Definition. Let $\mathcal{V}'$ be the complement of $\mathcal{Z}_I$ and $\mathcal{Z}_J$ in $\mathcal{V}$:

$$\mathcal{V}' = \mathcal{V} - (\mathcal{Z}_I \cup \mathcal{Z}_J). \tag{20}$$

The intersections of level sets LSV_I and LSV_J with $\mathcal{V}'$ are denoted LSV'_I and LSV'_J, respectively.

5.1. Geometry of the level sets in $\mathcal{V}'$

$\mathcal{V}'$ is a union of 3-dimensional torus orbits that are parsed in different ways by the families of varieties LSV'_I and LSV'_J. An orbit that belongs to the generic stratum lies in a general level set of each fibration. On the nongeneric orbits, certain Plücker coordinates vanish. Those on which π_i or π_i^* vanishes belong to a special level set LSV'_I, and those on which some π_{ij} vanishes belong to a special level set LSV'_J. Here the polytope is missing the vertices of a hexagonal or a square face, respectively, or both if it belongs to a special level set of both I and J. The special level sets of I and J in $\mathcal{V}'$ are the subvarieties $\pi_i\pi_i^* = 0$ and $\pi_{ij}\pi_{kl} = 0$, respectively.

Definition. Let $H\backslash\mathrm{LSV}'_I$ be the quotient by the torus action of a level set of I in $\mathcal{V}'$, and let $f_J : H\backslash\mathrm{LSV}'_I \to \mathbf{P}^1$ be the map that sends a torus orbit to its value of J.

Proposition 5.1.

1. *If LSV'_I is general, then $H\backslash LSV'_I$ is isomorphic to $\mathbf{P}^1$, and $f_J : H\backslash LSV'_I \to \mathbf{P}^1$ is two-to-one, branched over two distinct points.*
2. *Let LSV'_I be the special level set with $I_{21} = -\lambda_i^2$. Then LSV'_I is the subvariety of $\mathcal{V}'$ cut out by $\pi_i\pi_i^* = 0$, and $H\backslash LSV'_I$ consists in two copies of $\mathbf{P}^1$ that intersect in one point. The map $f_J : H\backslash LSV'_I \to \mathbf{P}^1$ is bijective on each component.*

Proof. A general level set LSV'_I in $\mathcal{V}'$ is a 1-parameter family of torus orbits,

$$\mathrm{LSV}'_I = \Pi_1^{-1}(\mathcal{O}), \tag{21}$$

where $\mathcal{O}$ is generic in G/P_1. Since J is a linear fractional transformation of the cross-ratio, part (1) follows from Proposition 3.2. From (9) and Proposition 3.3, a special level set has the form

$$\mathrm{LSV}'_I = [\Pi_1^{-1}(\mathcal{O}^i) - \mathcal{O}_{\diamond(\backslash i^*)^c}] \cup [\Pi_1^{-1}(\mathcal{O}^{i^*}) - \mathcal{O}_{\diamond(\backslash i)^c}] \cup \tilde{\mathcal{O}}^{ii^*}, \tag{22}$$

where $\mathcal{O}_{\diamond(\backslash i^*)^c}$ and $\mathcal{O}_{\diamond(\backslash i)^c}$ belong to the base locus $\mathcal{Z}_J$. This is the reducible subvariety of $\mathcal{V}'$ cut out by $\pi_i\pi_i^* = 0$; it consists in two components that intersect in the single orbit $\tilde{\mathcal{O}}^{ii^*}$. □

The J-fibration of $\mathcal{V}'$ is completely analogous.

Definition. Let $H\backslash\mathrm{LSV}'_J$ be the quotient by the torus action of a level set of J in $\mathcal{V}'$, and denote by $f_I : H\backslash LSV'_J \to \mathbf{P}^1$ the map that sends a torus orbit to its value of I_{21}.

Proposition 5.2.

1. *If LSV'_J is a general level set with $J \neq 0$, then $H\backslash LSV'_J$ is isomorphic to $\mathbf{P}^1$, and $f_I : H\backslash LSV'_J \to \mathbf{P}^1$ is two-to-one, branched over two distinct points.*
2. *Let LSV'_J be the special level set with $J = -(\lambda_i + \lambda_j)^2$. Then LSV'_J is the subvariety of $\mathcal{V}'$ cut out by $\pi_{ij}\pi_{kl} = 0$, and $H\backslash LSV'_J$ consists in two components, each isomorphic to $\mathbf{P}^1$, which intersect in one point. The map f_I is bijective on each component.*
3. *When $J = 0$, $H\backslash LSV'_J$ consists in two components, each isomorphic to $\mathbf{P}^1$, that intersect in one point. The map f_I is bijective on each component and takes the value ∞ at the point of intersection.*

First, observe the structure of the level sets LSV'_J in $\mathcal{V}'$. A general level set is a 1-parameter family of torus orbits

$$\mathrm{LSV}'_J = \Pi_2|_{\mathcal{V}}^{-1}(\mathcal{Q}), \tag{23}$$

where $\mathcal{Q}$ is generic in $Gr(2,4)$, and a special level set has the form

$$\mathrm{LSV}'_J = [\Pi_2|_{\mathcal{V}}^{-1}(\mathcal{Q}^{ij}) - \mathcal{Q}_{\diamond(\backslash kl)^c}] \cup [\Pi_2|_{\mathcal{V}}^{-1}(\mathcal{Q}^{kl}) - \mathcal{Q}_{\diamond(\backslash ij)^c}] \cup \tilde{\mathcal{Q}}^{(ij)(kl)} \tag{24}$$

(cf. Proposition 4.3). This is the subvariety of $\mathcal{V}'$ cut out by $\pi_{ij}\pi_{kl} = 0$. It consists of two components that intersect in the single orbit $\tilde{\mathcal{Q}}^{(ij)(kl)}$. To prove the proposition, we consider the following elliptic curve.

5.2. The elliptic curve of branch points

The set of pairs (I_{21}, J) such that J is a branch point of $f_J : H\backslash\mathrm{LSV}'_I \to \mathbf{P}^1$ defines an elliptic curve in $\mathbf{P}^1 \times \mathbf{P}^1$. To see this, consider the quadratic polynomial (17). For a fixed general value of I_{21} and a fixed value $J = J_0$, there are two values of u, counting multiplicity, for which it vanishes. These values of u cut out the two orbits (or single orbit) in LSV'_I with $J = J_0$. The two values of J over which f_J is branched are those for which the discriminant of (17) as a polynomial in u vanishes. This discriminant is given in (18).

For a general level set ($I_{21} \neq -\lambda_i^2$), the two values of J for which it vanishes are distinct, and for the special level set $I_{21} = -\lambda_i^2$, they coincide at $J = s_3/(2\lambda_i)$. Writing (18) in homogeneous coordinates $[I_{21} : X] \times [J : Y]$ gives a homogeneous polynomial of degree four:

$$I_{21}^2 JY + I_{21}XJ^2 + s_2 I_{21}XJY - s_4X^2JY - \tfrac{1}{4}s_3^2X^2Y^2. \tag{25}$$

For $s_3 \neq 0$, its vanishing set is a nonsingular elliptic curve in $\mathbf{P}^1 \times \mathbf{P}^1$, denoted as $\mathcal{E}$. Indeed, the projection of $\mathcal{E}$ onto the coordinate $[I_{21} : X]$ realizes it as a double cover of $\mathbf{P}^1$ branched over the four values $[-\lambda_i^2 : 1]$.

Proof of Proposition 5.2. Consider the variety $H\backslash \mathrm{LSV}'_J$ in $H\backslash \mathcal{V}'$. For any J, $f_I : H\backslash \mathrm{LSV}'_J \to \mathbf{P}^1$ is a 2:1 surjective map (by Proposition 4.3 for special values of J and by (17) and Proposition 4.2 for general J). For fixed $[J : Y]$, $\mathcal{E}$ defines a quadratic polynomial in I_{21} whose roots are the values of I_{21} over which f_I is branched. The values of J where the two roots coincide are those for which the discriminant, (19), of (25) as a polynomial in I_{21} vanishes. These values are $J = 0, -(\lambda_1 + \lambda_2)^2, -(\lambda_1 + \lambda_3)^2$, and $-(\lambda_1 + \lambda_4)^2$. Since $s_3 \neq 0$, they are distinct, and the three values $J = -(\lambda_1 + \lambda_j)^2$ correspond to the special level sets of J.

For J general and not zero, f_I has two branch points of order 1. So by the Riemann–Hurwitz formula, LSV'_J is $\mathbf{P}^1$. By Proposition 4.3 and (24), a special level set LSV'_J consists of two components, each parametrized by I_{21}, which intersect in the point corresponding to $\tilde{Q}^{(ij)(kl)}$. By (16), $H\backslash \mathrm{LSV}'_{J=0}$ also consists of two components, $u = \infty$ and $u = (I_{21}^2 + s_2 I_{21} - s_4)/s_3$. Each is a copy of $\mathbf{P}^1$, and they intersect in the unique point with $I_{21} = u = \infty$. (Indeed, putting $J = 0$ in (25) gives the solution $I_{21} = \infty$ with multiplicity two.) □

5.3. *Monodromy of the I_{21}-fibration*

Each level set of I_{21} in $\mathcal{V}'$ contains orbits whose points are fixed by the Hamiltonian flow of I_{21}. This flow is obtained by factoring $e^{t\nabla I_{21}(X_0)}$ as in (2) and conjugating X_0 by $n(t)$ as in (3). The gradient is computed in the whole algebra $sl(4, \mathbf{C})$ with respect to the Killing form and then evaluated on $\epsilon + \mathcal{B}_-$ [3]. The fixed points are those where ∇I_{21} is upper triangular, and this occurs where $g_2k - h_1h_2 = 0$. On the matrices (15), this condition is equivalent to $I_{21} + u^2 = 0$, and by (16), it occurs where $J = \infty$. Each level set LSV'_I with $I_{21} \neq 0, \infty$ contains two such orbits, cut out by $u = \pm\sqrt{-I_{21}}$. For $I_{21} = 0, \infty$, these values coincide.

Removing the two orbits of I_{21}-fixed points from a generic LSV'_I produces a variety generated by the $(\mathbf{C}^*)^3$-action of the diagonal torus, together with a $\mathbf{C}^*$-action of the I_{21}-flow. Taking the quotient of $\mathcal{V}' - \{J = \infty\}$ by the torus action therefore yields a $\mathbf{C}^*$-fiber bundle with singular fibers, where each fiber is generated by the Hamiltonian flow of I_{21}.

Proposition 5.3. *Let $\bar{f}_I : H\backslash(\mathcal{V}' - \{J = \infty\}) \to \mathbf{P}^1$ be the map that sends each point to its value of I_{21}. The generic fiber of $\bar{f}_I$ is isomorphic to $\mathbf{C}^*$. If $\lambda_i \neq 0$ for all i, then there* *are six singular fibers; they lie over* $I_{21} = -\lambda_1^2, -\lambda_2^2, -\lambda_3^2, -\lambda_4^2, 0,$ *and* ∞. *The fiber over* $-\lambda_i^2$ *has two components, each isomorphic to* $\mathbf{C}$, *which intersect in one point. The fibers over zero and infinity are each isomorphic to* $\mathbf{C}$.

Proof. By (16), when $I_{21} \neq -\lambda_i^2$, $H\backslash \mathrm{LSV}'_I$ contains two points with $J = \infty$ if $I_{21} \neq 0, \infty$ and a single such point if $I_{21} = 0, \infty$. Since $H\backslash \mathrm{LSV}'_I$ is isomorphic to $\mathbf{P}^1$, removing the fixed points of the I_{21}-flow produces a fiber isomorphic to $\mathbf{C}^*$ if $I_{21} \neq 0, \infty$ and a fiber isomorphic to $\mathbf{C}$ if $I_{21} = 0, \infty$.

When $I_{21} = -\lambda_i^2$, $H\backslash \mathrm{LSV}'_I$ consists in two copies of $\mathbf{P}^1$ that intersect in one point. Since f_J takes the value $s_3/(2\lambda_i) \neq \infty$ at the point of intersection, there are two distinct points in $H\backslash \mathrm{LSV}'_I$ with $J = \infty$, one in each component. Removing these two points yields two copies of $\mathbf{C}$ that intersect in the single point corresponding to the orbit $\tilde{\mathcal{O}}^{ii*}$. □

To determine the monodromy near the singular fiber $I_{21} = -\lambda_i^2 \neq 0$, let $\mathcal{U}$ be a neighborhood of this fiber in $H\backslash(\mathcal{V}' - \{J = \infty\})$ that contains no other singular fiber. The map f_J, restricted to a generic fiber, realizes it as a two-to-one cover of $\mathbf{P}^1 - \{\infty\}$ branched over two distinct points. These points come together as I_{21} approaches $-\lambda_i^2$ and coincide in the limit. This behavior of the branch points can be seen in the projection $\phi_I : \mathcal{E} \mapsto \mathbf{P}^1$ from the elliptic curve of branch points to the coordinate $[I_{21} : X]$.ϕ_I is itself a two-to-one cover of $\mathbf{P}^1$ branched over the four values $I_{21} = -\lambda_k^2$. In a neighborhood of a branch point, there is a local complex coordinate z on $\mathcal{E}$ where $\phi_I(z) = w = z^2$. This map is branched over $w = 0$. As w traverses the path $w(\theta) = \mathrm{e}^{\mathrm{i}\theta}, 0 \leq \theta \leq 2\pi$, the two points $\phi_I^{-1}(\mathrm{e}^{\mathrm{i}\theta}) = \{\pm \mathrm{e}^{\mathrm{i}\theta/2}\}$ rotate counterclockwise halfway around zero, interchanging places.

This is the same behavior that occurs near the singular fiber $g^{-1}(0)$ in the $\mathbf{C}^*$-fibration of $\mathbf{C}^2$ by the level sets of $g(u, v) = u^2 + v^2$. Arnold [1] finds that the monodromy of this fibration near the singular fiber produces a single twist of the noncompact cycle in $\mathbf{C}^*$ around the $\mathbf{C}^*$-cylinder.

The singularity at $I_{21} = 0$ is of a different type and has a different monodromy. Recall that in a special level set of I_{21} in $H\backslash\mathcal{V}'$, the two branch points of the double cover $f_J : H\backslash \mathrm{LSV}'_I \mapsto \mathbf{P}^1$ coincide, and the two points with $J = \infty$ are distinct. In the level set $I_{21} = 0$, the situation is reversed. By (18), the two branch points, $J = -s_3^2/(4s_4)$ and $J = \infty$, of $\bar{f}_J$ are distinct. (Here we assume that $s_4 \neq 0$ so that no λ_i is zero. If $s_4 = 0$, then the level set $I_{21} = 0$ would consist in two components, and the two types of singularities would coincide in the same level set.) On the other hand, the two fixed points of the I_{21}-flow in $H\backslash \mathrm{LSV}_{I_{21}=0}$ coincide since $u = 0$ is a solution of multiplicity two of $I_{21} + u^2 = 0$ when $I_{21} = 0$. This behavior occurs in the singular fiber of the $sl(2, \mathbf{C})$ Toda lattice. The monodromy, described in [9], produces a double twist of the noncompact cycle around $\mathbf{C}^*$. This corresponds to a reversal of the two fixed points of the flow, one a source and the other a sink.

In $sl(4, \mathbf{C})$, this monodromy is seen explicitly through the following isomorphism between a neighborhood of the fiber $I_{21} = 0$ and a neighborhood of the singular fiber in the $sl(2, \mathbf{C})$ Toda lattice. In [9], the phase space for $sl(2, \mathbf{C})$ is embedded into $\mathbf{P}^1 \times \mathbf{P}^1$ by the map

$$\begin{pmatrix} a & 1 \\ b & -a \end{pmatrix} \mapsto [a^2 + b : 1] \times [a : 1].$$

The constant of motion is $\gamma = a^2 + b$, and the two fixed points occur where $a = \pm\sqrt{\gamma}$. They coincide in the singular level set $\gamma = 0$. The $\mathbf{C}^*$-bundle of level sets, $\mathcal{M}$, is obtained from $\mathbf{P}^1 \times \mathbf{P}^1$ by removing the two fixed points from each generic level set, together with the level sets $\gamma = 0$ and ∞.

Let $\mathcal{U} \subset H\backslash(\mathcal{V}' - \{J = \infty\})$ be an open set containing the degenerate fiber $I_{21} = 0$ such that $\mathcal{U}$ contains no other singular fiber. We may take it to be a union of fibers. The points in a $\mathbf{C}^*$-fiber of $H\backslash(\mathcal{V}' - \{J = \infty\})$ for a fixed general value of I_{21} are parametrized by $u \in \mathbf{P}^1 - \{\pm\sqrt{-I_{21}}\}$ and are represented by the matrices $X(I_{21}, u)$ in (15) when $u \neq \infty$. The mapping

$$g : \mathcal{U} \to \mathcal{M},$$

$$X(I_{21}, u) \mapsto \begin{pmatrix} u & 1 \\ -(I_{21} + u^2) & -u \end{pmatrix}, \tag{26}$$

where the image matrix corresponds to $[-I_{21} : 1] \times [u : 1]$ in $\mathbf{P}^1 \times \mathbf{P}^1$, defines an isomorphism between $\mathcal{U}$ and its image when extended to include $u = \infty$. The monodromy near the singular fiber in $\mathcal{U}$ is therefore also a double twist of the noncompact cycle in $\mathbf{C}^*$. In this case, since the $\mathbf{C}^*$-fiber is generated by the Hamiltonian flow of I_{21}, the monodromy reflects a reversal of the source and the sink of the I_{21}-flow. The same singular behavior occurs in a neighborhood of the fiber $I_{21} = \infty$.

5.4. Monodromy of the J-fibration

The fixed points of the J-flow occur where ∇J is upper triangular. These points are all contained in the level set $J = 0$. On the other hand, the variety $I_{21} = \infty$ typically intersects a level set of J in two torus orbits. It does not correspond to a generic symplectic leaf in $\epsilon + \mathcal{B}_-$. We remove it from $\mathcal{V}'$ to obtain a $\mathbf{C}^*$-bundle over P^1 whose fibers are parametrized by J.

Proposition 5.4. *Let $\bar{f}_J : H\backslash(\mathcal{V}' - \{I_{21} = \infty\}) \to \mathbf{P}^1$ be the map that sends a point to its value of J. Its generic fiber is isomorphic to $\mathbf{C}^*$. There are five singular fibers, which lie over $J = -(\lambda_1 + \lambda_2)^2, -(\lambda_1 + \lambda_3)^2, -(\lambda_1 + \lambda_4)^2, 0,$ and ∞. The fiber over $J = -(\lambda_1 + \lambda_i)^2$ consists of two components, each isomorphic to $\mathbf{C}$, that intersect in one point. The fiber over $J = \infty$ is isomorphic to $\mathbf{C}$, and the fiber over $J = 0$ consists of two disjoint copies of $\mathbf{C}$.*

Proof. By (25), LSV'_J contains two distinct points with $I_{21} = \infty$ if $J \neq 0, \infty$ and one such point if $J = 0, \infty$, and by Proposition 5.2, LSV'_J is isomorphic to $\mathbf{P}^1$ when $J \neq -(\lambda_i + \lambda_j)^2$ and $J \neq 0$. The fiber of $\bar{f}_J$ in $H\backslash(\mathcal{V}' - \{I_{21} = \infty\})$ is therefore isomorphic to $\mathbf{C}^*$ if $J \neq 0, \infty$, or $-(\lambda_i + \lambda_j)^2$. Since $H\backslash\mathrm{LSV}'_{J=\infty}$ is isomorphic to $\mathbf{P}^1$, removing a single point produces a fiber isomorphic to **C**. In $H\backslash \mathrm{LSV}'_{J=0}$, I_{21} is ∞ at the unique point of intersection of two copies of $\mathbf{P}^1$, resulting in two disjoint copies of **C** (Proposition 5.2(3)).

When $J = -(\lambda_i + \lambda_j)^2$, $H\backslash \mathrm{LSV}'_J$ consists in two copies of $\mathbf{P}^1$ that intersect in the orbit $\tilde{Q}^{(ij)(kl)}$. Since the value of I_{21} on this orbit is $(1/2)(\lambda_i\lambda_j + \lambda_k\lambda_l) \neq \infty$, there are two distinct orbits in LSV'_J with $I_{21} = \infty$, one in each component. Removing these two orbits gives a fiber isomorphic to two copies of **C** that intersect in one point. □

Near a special level set of J in $H\backslash(\mathcal{V}' - \{I_{21} = \infty\})$, the structure of the $\mathbf{C}^*$-fibration is the same as that of the $\mathbf{C}^*$-fibration of $H\backslash(\mathcal{V}' - \{J = \infty\})$ near a special level set of I_{21}. The monodromy near each of the singular fibers $J = -(\lambda_1 + \lambda_j)^2$ twists the noncompact cycle once around $\mathbf{C}^*$. The singularity at $J = \infty$ is the same type that occurs in the fiber $I_{21} = 0$ in $H\backslash(\mathcal{V}' - \{J = \infty\})$, where the monodromy is a double twist of the noncompact cycle around $\mathbf{C}^*$.

6. A twofold symmetry

The fact that a pair of level sets LSV'_I and LSV'_J typically intersect in two distinct torus orbits is a reflection of a symmetry of order two in $\epsilon + \mathcal{B}_-$ that occurs for general $sl(n, \mathbf{C})$. It is induced by the automorphism group $\mathbf{Z}_2$ of the Dynkin diagram of $sl(n, \mathbf{C})$ [10]. In terms of X in $\epsilon + \mathcal{B}_-$, the symmetry is expressed in the involution

$$\sigma(X) = \eta X^t \eta,$$

where η is the matrix with 1's on the antidiagonal and 0's elsewhere. This mapping is reflection across the antidiagonal.

It is shown in [10] that the involution preserves each k-chop integral I_{rk} and that under the torus embedding, σ induces an involution on the quotient of the flag manifold by the torus action. Indeed, for $h \in (\mathbf{C}^*)^3$,

$$\sigma \circ h = h^{-1} \circ \sigma, \tag{27}$$

where σ denotes the induced involution on the image of X in G/B. By direct calculation, one can check that σ also preserves J. σ therefore induces an involution on the set of 3-dimensional torus orbits in each of the level sets LSV'_I and LSV'_J. We will show that σ interchanges the two orbits in which a typical pair LSV'_I and LSV'_J intersect. To do this, we first show that σ intertwines the action of the I_{21}-flow in the same way that it intertwines the flows of the three basic Hamiltonians in (27).

In $\epsilon + \mathcal{B}_-$, the flow of I_{21} is obtained by conjugating X by matrices of the form

$$Z = Z(u) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1-u & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad u \in \mathbf{C}.$$

Lemma 6.1. *Let* $Z(X) = ZXZ^{-1}$ *for* $X \in \epsilon + \mathcal{B}_-$. *Then* $(\sigma \circ Z)(X) = (Z^{-1} \circ \sigma)(X)$.

$$
\begin{aligned}
(\sigma \circ Z(X) &= \eta(ZXZ^{-1})^t\eta \\
&= \eta(ZLV\Lambda V^{-1}L^{-1}Z^{-1})^t\eta \\
&= \eta(Z^{-1})^t(L^{-1})^t(V^{-1})^t\Lambda V^tL^tZ^t\eta \\
&= Z^{-1}\eta(L^{-1})^t(V^{-1})^t\Lambda V^tL^t\eta Z \\
&= Z^{-1}\eta X^t\eta Z \\
&= (Z^{-1} \circ \sigma)(X).
\end{aligned}
$$

Proposition 6.1. *If* LSV'_I *contains two torus orbits with* $J = J_0$, *then* σ *interchanges these orbits; otherwise* σ *preserves the (unique) orbit with* $J = J_0$.

The corresponding statement obtained by interchanging I and J follows from this. Proposition 6.1 therefore implies that σ interchanges the two orbits in which a level set of I_{21} and a level set of J in $\mathcal{V}'$ intersect.

Proof. From (16), J has the value zero when $u = \infty$. We will prove the statement for $J_0 \neq 0$, and then it will follow for $J_0 = 0$ by the continuity of the induced involution on $H\backslash \mathrm{LSV}'_I$.

If $f_J : H\backslash \mathrm{LSV}'_I \to \mathbf{P}^1$ is branched over J_0, then since σ preserves J, it preserves the unique orbit with $J = J_0$.

Now consider a general level set LSV'_I, and suppose f_J is not branched over J_0. Let β be one of the two branch points of f_J (we may assume $f_J(\beta) \neq 0$), and let ξ_1 and ξ_2 be the two distinct points in $H\backslash \mathrm{LSV}'_I$ with $J = J_0$. Since σ preserves $J, \sigma(\xi_1) = \xi_1$ or $\sigma(\xi_1) = \xi_2$, and $\sigma(\beta) = \beta$. There is a unique value u such that $Z(u)\xi_1 = \beta$ (here $Z(u)\xi_1$ refers to the action of $Z(u)$ on $H\backslash \mathrm{LSV}'_I$ induced by its action on $\epsilon + \mathcal{B}_-$). By the lemma, $\sigma \circ Z(u) = Z^{-1}(u) \circ \sigma$. So if $\sigma(\xi_1) = \xi_1$, then $\beta = \sigma(\beta) = (\sigma \circ Z(u))\xi_1 = (Z^{-1}(u) \circ \sigma)\xi_1 = Z^{-1}(u)\xi_1 = Z(2-u)\xi_1$. But this is a contradiction since $2 - u \neq u$. So $\sigma(\xi_1) = \xi_2$. □

In the special level set where $I_{21} = -\lambda_i^2$, the polytopes of the two orbits with $J = J_0$ are $\diamond(\backslash i)$ and $\diamond(\backslash i^*)$. σ interchanges these two orbits since it induces the antipodal map on the vertices of the moment polytope [10].

7. Generalization to other generic leaves

The geometry in the preceding sections extends to all the generic leaves. For an arbitrary fixed value of the Casimir I_{11}, the characteristic polynomial of $\phi_1(X)$ is $\lambda^2 + I_{11}\lambda + I_{21} = 0$, and the special values of I_{21} are $-\lambda_i^2 - I_{11}\lambda_i$. When these are distinct, the variety $\mathcal{V}$, defined by (6) with this value of I_{11} contains four special level sets of type I.

There is a generalization of (15) for general I_{11}:

$$X(I_{11}, I_{21}, u) = Z(u)LC_\Lambda L^{-1}Z^{-1}(u)$$

$$= \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & u & 1 & 0 \\ 0 & -(u^2 + I_{11}u + I_{21}) & -u - I_{11} & 1 \\ s_4 - s_3 + s_2 - 1 & h_2(I_{11}, I_{21}, u) & g_3(I_{11}, I_{21}, u) & 1 + I_{11} \end{pmatrix},$$

where

$$L^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ -1 & -I_{21} - I_{11} & -1 - I_{11} & 1 \end{pmatrix},$$

and $Z^{-1}(u)$ is as before. This yields

$$J(I_{11}, I_{21}, u) = -\frac{(I_{21}^2 + s_2 I_{21} - s_3 u - s_4) + I_{11}(2I_{21}u + s_2 u) + I_{11}^2 u^2}{u^2 + I_{11}u + I_{21}},$$

which reduces to (16) when $I_{11} = 0$. Calculations using MAPLE verify that for arbitrary I_{11}, J is equal to the linear fractional transformation (11) of the cross-ratio. The special level sets of J therefore occur at the same values as before.

In a level set of I_{21}, the orbits that are fixed by the I_{21}-flow occur at the values of u where $u^2 + I_{11}u + I_{21} = 0$. These values coincide when $I_{21} = I_{11}^2/4$. In calculating the monodromy near this level set, the mapping (26) becomes

$$X(I_{11}, I_{21}, u) \mapsto \begin{pmatrix} u + I_{11}/2 & 1 \\ -(u^2 + I_{11}u + I_{21}) & -u - I_{11}/2 \end{pmatrix}.$$

This defines an isomorphism between a neighborhood of the singular fiber $I_{21} = I_{11}^2/4$ in the $\mathbf{C}^*$-bundle $H\backslash(\mathcal{V}' - \{J = \infty\})$, where $\mathcal{V}$ now means the variety where the Casimir has the fixed value I_{11}, and a neighborhood of the singular fiber $\gamma = 0$ in the $\mathbf{C}^*$-bundle of $sl(2, \mathbf{C})$.

8. Conclusion

The closure of a generic leaf in $\epsilon + \mathcal{B}_-$ contains leaves of dimensions 6, 4, 2, and 0. In [11] it is shown that the two fixed points in the $\mathbf{P}^1$ of 3-dimensional orbits in a typical LSV_I correspond to isospectral sets in leaves of dimension 6. The 3-dimensional orbits in the base loci of the varieties LSV_I and LSV_J and those where $I = \infty$ or $J = \infty$ correspond to 6-dimensional leaves of a different kind and the orbits of dimension less than three to leaves of dimension less than 6.

In $sl(4, \mathbf{C})$, where there is only one Casimir and one constant of motion in addition to the basic Hamiltonians, it is easy to work with the polytopes and the flows in the flag manifold explicitly. When $n > 4$, the picture is more complicated because of the Casimirs and constants of motion beyond the 1-chop level. In $sl(6, \mathbf{C})$, for example, there are two Casimirs and four additional integrals — three 1-chop integrals and a 2-chop integral. Their typical level set in G/B is the closure of a four-parameter family of 5-dimensional orbits, and it no longer contains the complete inverse image of a generic torus orbit in G/P_1. The k-chop integrals with $k > 2$ depend on additional partial flag manifolds; for general $sl(n, \mathbf{C})$, a single level set variety is conjugate to a product of torus orbits in a product of partial flag manifolds of decreasing dimension [3].

The full Kostant–Toda lattice has multiple involutive families of integrals. These may be constructed (abstractly) by building families of invariants along different nested chains of parabolic subalgebras of $sl(n, \mathbf{C})$ as described in ([3,12]). When $n = 4$, one can use the special isomorphism $sl(4, \mathbf{C}) \cong so(6, \mathbf{C})$ to calculate the integral J explicitly. In the flag manifold, the geometry of the J-fibration of level sets is essentially obtained by replacing the partial flag manifold $\{V^1 \subset V^3 \subset \mathbf{C}^4\}$ in the description of the I-fibration by the Grassmannian of 2-planes in $\mathbf{C}^4$. This suggests that in higher-dimensional Toda lattices, different involutive families of integrals may be associated to different partial flag manifolds.

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